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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

Noncommutative differential direct Lie derivative and the algebra of special Euclidean group $SE(2)$

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ABSTRACT. A Lie algebra is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds. The term "Lie algebra"(after Sophus Lie) was introduced by Hermann Weyl in the 1930s. Using the Lie algebra of the special euclidean group $SE(2)$ and using notions of noncommutative geometry, we can construct a new derivative which I call it direct Lie derivative. Also in this paper we define an real inner product for A,B? $se(2)$ two elements of the Lie algebra $se(2)$. The main result of this paper consist in the proof that we can generate other Lie algebras using the direct Lie derivative elements.

1. INTRODUCTION

Let

$$SO(2) = \left\{ A \in M_{2,2}(\mathbb{R}) \mid A^T A = I_2, \det(A) = 1 \right\}$$

Using the definition of Euclidean groups, one obtains

$$SE(2) = SO(2) \times \mathbb{R}^2$$

Let us remind some basic properties of $SE(2)$.

Proposition 1.1. *$SE(2)$ can be canonically identified with a $SL(2,R)$ subgroup*

$$SE(2) = \left\{ \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix} \mid A \in SO(2), a \in \mathbb{R}^2 \right\}$$

Proposition 1.2. *$SE(2)$ can be canonically identified with*

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi] \right\}$$

Proposition 1.3. *$SE(2)$ has a Lie group structure and as a Lie algebra has the following form*

$$se(2) = \left\{ \begin{pmatrix} 0 & -a & v_1 \\ a & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix} \mid a, v_1, v_2 \in \mathbb{R} \right\}$$

For the algebra of $SE(3)$, given by

$$se(3) = \left\{ \begin{pmatrix} \widehat{w} & u \\ 0 & 0 \end{pmatrix} \in GL(4) \mid \widehat{w} \in so(3), u \in \mathbb{R}^3 \right\}$$

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we know the "Rodrigues" formula

$$e^{\widehat{w}\theta} = I_3 + \widehat{w}\sin\theta + \widehat{w}^2(1-\cos\theta)$$

where

$$\widehat{w} = \begin{pmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{pmatrix}$$

Here $\widehat{w}$ represents the skew symmetric matrix form of the rotation vector $w = (w_x, w_y, w_z)^T$ and it is an element from the Lie algebra of $SO(3)$.

The exponential map

$$\exp : se(3) \to SE(3)$$

is well defined and surjective and is given by

$$\exp\begin{pmatrix} \widehat{w} & u \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp(\widehat{w}) & Au \\ 0 & 0 \end{pmatrix}$$

where

$$A = I_3 + \frac{1-\cos(\| w \|)}{\| w \|^2}\widehat{w} + \frac{\| w \| - \sin(\| w \|)}{\| w \|^3}\widehat{w}^2$$

or the algebra $se(2)$, we know (see [1])

$$\exp A = \begin{cases} I_3 + \frac{\sin a}{a}A + \frac{1-\cos a}{a^2}A^2, a \neq 0 \\ \begin{pmatrix} 1 & 0 & v_1 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{pmatrix}, a = 0 \end{cases}$$

with $A = \begin{pmatrix} 0 & -a & v_1 \\ a & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix}$.

We can define a real inner product $(\cdot \mid \cdot)$ on $se(2)$, by

$$(A \mid B) = -Tr(A \cdot B)$$

where $A, B \in se(2)$.

Also, if we take $A = \begin{pmatrix} 0 & -a & v_1 \\ a & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -b & u_1 \\ b & 0 & u_2 \\ 0 & 0 & 0 \end{pmatrix}$, one obtains $(A \mid B) = -Tr(A \cdot B) = 2ab$.

For the Lie group $SO(3)$ and $SU(2)$ we will present some general properties. The Lie algebra associated for this two Lie groups are $so(3), su(2)$ and having the following bases

$$so(3) : P = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$su(2) : H = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, F = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The non trivial Lie brackets in this case, are

$$[P, Q] = R, [Q, R] = P, [R, P] = Q$$

$$[H,E]=F,[E,F]=H,[F,H]=E$$

This implies the isomorphism

$$\varphi : su(2) \to so(3)$$

$$\varphi(xH+yE+zF)=xP+yQ+zR$$

where $x,y,z \in \mathbb{R}$.

This isomorphism satisfies

$$\varphi\left([U,V]\right)=\left([\varphi(U),\varphi(V)]\right)$$

and so is an Lie algebra isomorphism.

2. Main results

In noncommutative geometry for an R-algebra A, exists one derivation

$$d : A \to \Omega_{na}(A)$$

(here the symbol *na* means non-abelian and $\Omega_{na}(A)$ is an bi-module I).

Here, *I* represents the ker of the application

$$\mu : A \otimes_R A \to A$$

and *A* is an R-algebra.

If we identified the algebra *A* with the Lie algebra $se(2)$, one obtains

$$\begin{array}{lcl} d & : & se(2) \to \Omega_{na}(se(2)) \\ f & \to & 1 \otimes f - f \otimes 1 \end{array}$$

After simple computations, one obtains

$$df = 1\otimes f - f \otimes 1 = \begin{pmatrix} 0 & -a & v_1 & a & 0 & 0 & -v_1 & 0 & 0 \\ a & 0 & v_2 & 0 & a & 0 & 0 & -v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & -v_1 \\ -a & 0 & 0 & 0 & -a & v_1 & -v_2 & 0 & 0 \\ 0 & -a & 0 & a & 0 & v_2 & 0 & -v_2 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & -v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In this way we can define a derivative which we will call it ”the direct Lie derivative”.

Let

$$df, dg \in \Omega_{na}(se(2))$$

with the following form

$$df = \begin{pmatrix} 0 & -a & v_1 & a & 0 & 0 & -v_1 & 0 & 0 \\ a & 0 & v_2 & 0 & a & 0 & 0 & -v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & -v_1 \\ -a & 0 & 0 & 0 & -a & v_1 & -v_2 & 0 & 0 \\ 0 & -a & 0 & a & 0 & v_2 & 0 & -v_2 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & -v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a & v_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & v_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

$$dg = \begin{pmatrix} 0 & -b & u_1 & b & 0 & 0 & -u_1 & 0 & 0 \\ b & 0 & u_2 & 0 & b & 0 & 0 & -u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & -u_1 \\ -b & 0 & 0 & 0 & -b & u_1 & -u_2 & 0 & 0 \\ 0 & -b & 0 & b & 0 & u_2 & 0 & -u_2 & 0 \\ 0 & 0 & -b & 0 & 0 & 0 & 0 & 0 & -u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & u_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & u_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

Using this two elements df, dg we can proof the following lemma:

Lemma 2.1. *If $df, dg \in \Omega_{na}(se(2))$, for the Lie bracket we have*

$$[df, dg] \neq 0$$

Proof.

$$[df, dg] = df \cdot dg - dg \cdot df$$
$$\Longrightarrow [df, dg] \neq 0.$$

□

Proposition 2.4. *If $df, dg \in \Omega_{na}(se(2))$ we have*

1) $(df \mid df) = 0$;
2) $(df \mid dg) = 0$.

Proof. After easy computation, using the definition of the real inner product we can prove relation 1) and 2). □

Proposition 2.5. *If $df, dg, dh \in \Omega_{na}(se(2))$ then*

$$([df, dg] \mid dh) + (df \mid [dg, dh]) = 0$$

Proof. The above relation can be verified immediately using the Proposition 2.4 and Lemma 2.1. □

Now we can define the direct Lie derivative for the Lie algebra $so(3)$

$$\begin{array}{rcl} d & : & so(3) \to \Omega_{na}(so(3)) \\ f & \to & 1 \otimes f - f \otimes 1 \end{array}$$

Here, f represent the matrix $f = \begin{pmatrix} 0 & -a & u \\ a & 0 & -v \\ u & v & 0 \end{pmatrix}$. One obtains the derivative of the element f

$$df = 1 \otimes f - f \otimes 1 = \begin{pmatrix} 0 & -a & u & a & 0 & 0 & -u & 0 & 0 \\ a & 0 & -v & 0 & a & 0 & 0 & -u & 0 \\ u & v & 0 & 0 & 0 & a & 0 & 0 & -u \\ -a & 0 & 0 & 0 & -a & u & v & 0 & 0 \\ 0 & -a & 0 & a & 0 & -v & 0 & v & 0 \\ 0 & 0 & -a & u & v & 0 & 0 & 0 & v \\ -u & 0 & 0 & -v & 0 & 0 & 0 & -a & u \\ 0 & -u & 0 & 0 & -v & 0 & a & 0 & v \\ 0 & 0 & -u & 0 & 0 & -v & u & v & 0 \end{pmatrix}$$

Lemma 2.2. *If $df, dg, dh \in \Omega_{na}(so(3))$ we get for the Lie bracket*

$$[df, dg] \neq 0.$$

Proof.

$$[df, dg] = df \cdot dg - dg \cdot df \Longrightarrow [df, dg] \neq 0$$

and in this way the Lemma 2.2 is proved. □

Proposition 2.6. *If $df, dg, dh \in \Omega_{na}(se(2))$ and $\alpha, \beta \in \mathbb{R}$ we have*

$$[\alpha df + \beta dg, dh] = \alpha\,[df, dh] + \beta\,[dg, dh]$$

and

$$[dh, \alpha df + \beta dg] = \alpha\,[dh, df] + \beta\,[dh, dg]$$

Proof. Using the relations (2.1) and (2.2) one obtains $[\alpha df + \beta dg, dh]$ where

$$dh = \begin{pmatrix} 0 & -c & w_1 & c & 0 & 0 & -w_1 & 0 & 0 \\ c & 0 & w_2 & 0 & a & 0 & 0 & -w_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & -w_1 \\ -c & 0 & 0 & 0 & -c & w_1 & -w_2 & 0 & 0 \\ 0 & -c & 0 & c & 0 & w_2 & 0 & -w_2 & 0 \\ 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & -w_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & w_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & w_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Also, if we compute the expression $\alpha\,[df, dh] + \beta\,[dg, dh]$ we have

$$[\alpha df + \beta dg, dh] = \alpha\,[df, dh] + \beta\,[dg, dh]$$

□

Proposition 2.7. *If* $df, dg \in \Omega_{na}(se(2))$ *one obtains* $[df, dg] = -[dg, df]$

Proof.

$$[df, dg] = df \cdot dg - dg \cdot df = -(dg \cdot df - df \cdot dg) = -[dg, df]$$

□

Proposition 2.8. *If* $df, dg, dh \in \Omega_{na}(se(2))$ *we have*

$$[df, [dg, dh]] + [dg, [dh, df]] + [dh, [df, dg]] = 0$$

Proof.

$$[df, [dg, dh]] + [dg, [dh, df]] + [dh, [df, dg]] =$$
$$df \cdot dg \cdot dh - df \cdot dh \cdot dg - dg \cdot dh \cdot df + dh \cdot dg \cdot df + dg \cdot dh \cdot df - dg \cdot df \cdot dh$$
$$-dh \cdot df \cdot dg + df \cdot dh \cdot dg + dh \cdot df \cdot dg - dh \cdot dg \cdot df - df \cdot dg \cdot dh + df \cdot dg \cdot dh = 0$$

And so, the Proposition 2.8 is proved. □

In conclusion, we can generate other Lie algebras using the noncommutative direct Lie derivatives.

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