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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

n-Groups derivable from groups

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ABSTRACT. The trivial extentions of binary operation $*$ to the n-ary operation φ has the form $\varphi(x_1, x_2, \ldots, x_n) = x_1 * x_2 * \cdots * x_n * a$. If $(G, *)$ is a group and $a \in Z(G, *)$ the center of $(G, *)$ then (G, φ) is an n-group so called n-group derived from $(G, *)$. It is known that there exist n-groups (G, φ) which cannot be obtained as a derived group. The goal of the paper is to characterize all the n-groups operations which are derivable from group operations.

1. INTRODUCTION

In [3] M. Hosszú shows that every n-group (G, φ) can be obtained as an extention of a group $(G, *)$ using an automorphism α and an element $a \in G$. The n-ary operation corresponding is:

$$\varphi(x_1, x_2, \ldots, x_n) = x_1 * \alpha(x_2) * \cdots * \alpha^{n-1}(x_n) * a, \quad x_1, x_2, \ldots, x_n \in G$$

where $\alpha^n(x) = a * x * a^{-1}$, $x \in G$, $\alpha(a) = a$ and is denoted:

$$(G, \varphi) = Ext_{\alpha,a}(G, *).$$

The trivial extentions of groups where the n-ary operation has the form:

$$\varphi(x_1, x_2, \ldots, x_n) = x_1 * x_2 * \cdots * x_n * a, \quad x_1, x_2, \ldots, x_n \in G$$

where $a \in Z(G, *)$ the center of the group, are called n-groups derived from groups. This extentions are denoted

$$(G, \varphi) = Der_a(G, *).$$

In [2] W. A. Dudek and I. Michalski give an example of $(n+1)$-group (G, φ) which cannot be obtained as a derived group (there exists no group $(G, *)$ such that $(G, \varphi) = Der_a(G, *)$. The example is the following:

1.0. $G = \mathbb{Z}_{3^n-1}$, $\alpha : G \to G$, $\alpha(x) = 3x$, $x \in G$, $a = \frac{1}{2}(3^n - 1)$ and $(G, \varphi) = Ext_{\alpha,a}(G, +)$, that is:

$$\varphi(x_1, x_2, \ldots, x_n, x_{n+1}) = x_1 + 3x_2 + \cdots + 3^n x_{n+1} + \frac{1}{2}(3^n - 1),$$

$x_1, x_2, \ldots, x_n, x_{n+1} \in G$.

The goal of the paper is to characterize all the $(n+1)$-groups operations which are derivable from group operations.

We recall some notations and results which are used in the paper.

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1.1. If (G,φ) is a $(n+1)$-group and $u \in G$ an arbitrary element, then there exists an element $\overline{u} \in G$ with the property:

$$\varphi(x, \underset{k}{u}, \overline{u}, \underset{n-1-k}{u}) = \varphi(\underset{k}{u}, \overline{u}, \underset{n-1-k}{u}, x) = x \text{ for all } x \in G.$$

The element $\overline{u}$ is called the skew element of u.

1.2. If (G,φ) is a $(n+1)$-group then for all $u \in G$ the binary operation $\circ$: $G \times G \to G$, defined by:

$$x \circ y = \varphi(x, \underset{n-2}{u}, \overline{u}, y), \quad x, y \in G$$

determined on G a group structure $(G,\circ) = Red_u(G,\varphi)$, called reduced group in Hosszú sense. If we define

$$\alpha_u : G \to G \text{ by } \alpha_u(x) = \varphi(u, x, \underset{n-2}{u}, \overline{u}),\ x \in G \text{ and } a = \varphi(\underset{n+1}{u})$$

then $\alpha_u \in AutG$ (is an automorphism), $\alpha_u(a) = a$ and

$$(G,\varphi) = Ext_{\alpha_u,a}(G,*).$$

1.3. If $u, v \in G$ then between the reduced Hosszú groups $(G,\circ) = Red_u(G,\varphi)$ and $(G,*) = Red_v(G,\varphi)$ the following relations hold:

- $x * y = x \circ v' \circ y,\ x, y \in G$
- $\alpha_v(x) = v \circ \alpha_u(x) \circ \alpha_u(v'),\ x \in G$
- $v' = u * u$, is the inverse of v in the group $(G,\circ)$.

1.4. If (G,φ) is a $(n+1)$-group and $H \subset G$ is a nonempty subset, then (H,φ) is a sub-$(n+1)$-group in (G,φ) iff

a) $\varphi(u_1, u_2, \ldots, u_n, u_{n+1}) \in H$, for all $u_1, u_2, \ldots, u_n, u_{n+1} \in H$

b) $\overline{u} \in H$ for all $u \in H$.

2. Main results

Let (G,φ) be a $(n+1)$-group. We define the set $H = \{u \in G |\ \alpha_u = 1_G\}$, where $1_G : G \to G$ is the identity map of G.

We will show that if the set H is nonempty, then (H,φ) is a sub-$(n+1)$-group in (G,φ).

Lemma 2.1. *If $u \in H$ and $\overline{u}$ is the skew element of u, then $\overline{u} \in H$.*

Proof. We have

$$\alpha_u(x) = \varphi(u, x, \underset{n-2}{u}, \overline{u}) = x, \quad x \in G$$

and for $v = \overline{u}$

$$\alpha_{\overline{u}}(x) = \overline{u} \circ \alpha_u(x) \circ \alpha_u((\overline{u})') = \overline{u} \circ x \circ \alpha_u((\overline{u})') = \overline{u} \circ x \circ (\overline{u})'$$

It is enough to prove that $\overline{u} \circ x = x \circ \overline{u}$, $x \in G$ or $\varphi(\overline{u}, \underset{n-2}{u}, \overline{u}, x) = \varphi(x, \underset{n-2}{u}, \overline{u}, \overline{u})$.

But

$$\begin{aligned} \varphi(\overline{u}, \underset{n-2}{u}, \overline{u}, x) &= \varphi(\overline{u}, \underset{n-2}{u}, \varphi(\overline{u}, \underset{n-1}{u}, x), \overline{u}) = \\ &= \varphi(\overline{u}, \varphi(\underset{n-2}{u}, \overline{u}, \underset{2}{u}), \underset{n-3}{u}, x, \overline{u}) = \varphi(\overline{u}, u, \underset{n-3}{u}, x, \overline{u}) = \\ &= \varphi(\overline{u}, \underset{n-2}{u}, x, \overline{u}) = \varphi(\overline{u}, \underset{n-2}{u}, \alpha_u(x), \overline{u}) = \varphi(\overline{u}, \underset{n-2}{u}, \varphi(u, x, \underset{n-2}{u}, \overline{u}), \overline{u}) = \\ &= \varphi(\varphi(\overline{u}, \underset{n-2}{u}, u, x), \underset{n-2}{u}, \overline{u}, \overline{u}) = \varphi(x, \underset{n-2}{u}, \overline{u}, \overline{u}). \end{aligned}$$

□

Lemma 2.2. *If $u \in H$ and $x_1, x_2, \dots, x_n \in G$ then:*

$$\varphi(u, x_1, x_2, \dots, x_n) = \varphi(x_1, u, x_2, \dots, x_n) = \dots = \varphi(x_1, x_2, \dots, x_n, u).$$

Proof. $\varphi(u, x_1, x_2, \dots, x_n) = \varphi(u, \varphi(x_1, \underset{n-2}{u}, \overline{u}, u), x_2, \dots, x_n) =$

$$= \varphi(\varphi(u, x_1, \underset{n-2}{u}, \overline{u}), u, x_2, \dots, x_n) = \varphi(\alpha_u(x_1), u, x_2, \dots, x_n) =$$

$$= \varphi(x_1, u, x_2, \dots, x_n) = \varphi(x_1, u, \varphi(x_2, \underset{n-2}{u}, \overline{u}, u), \dots, x_n) =$$

$$= \varphi(x_1, \varphi(u, x_2, \underset{n-2}{u}, \overline{u}), u, \dots, x_n) = \varphi(x_1, \alpha_u(x_2), u, \dots, x_n) =$$

$$= \varphi(x_1, x_2, u, \dots, x_n) = \dots = \varphi(x_1, x_2, \dots, x_n, u).$$

□

Lemma 2.3. *If $u_1, u_2, \dots, u_n, u_{n+1} \in H$ then*

$$\varphi(u_1, u_2, \dots, u_n, u_{n+1}) \in H.$$

Proof. From Lemma 2.2 follows that

$$\varphi(u_1, u_2, \dots, u_n, u_{n+1}) = \varphi(u_{\sigma(1)}, u_{\sigma(2)}, \dots, u_{\sigma(n)}, u_{\sigma(n+1)})$$

for every $\sigma \in S_{n+1}$ (symmetric group).

If we denote $z = \varphi(u_1, u_2, \dots, u_n, u_{n+1})$, we have

$$\alpha_z(x) = \varphi(z, x, \underset{n-2}{z}, \overline{z})$$

and from [4]

$$\overline{z} = \varphi^{n^2-n-1}(\underset{n-2}{u_{n+1}}, \overline{u}_{n+1}, \underset{n-2}{u_n}, \overline{u}_n, \dots, \underset{n-2}{u_1}, \overline{u}_1)$$

and consequently

$$\alpha_z(x) = \varphi^{n^2}(x, \underbrace{\underset{n-1}{u_1}, \overline{u}_1}_{n-1}, \dots, \underbrace{\underset{n-1}{u_{n+1}}, \overline{u}_{n+1}}_{n-1}) = x, \quad x \in G.$$

□

From Lemma 2.1 and Lemma 2.3 we conclude the following theorem:

Theorem 2.1. *If (G, φ) is a $(n+1)$-group and the set*

$$H = \{u \in G |\ \alpha_u = 1_G\}$$

is nonempty, then (H, φ) is a sub-$(n+1)$-group in (G, φ).

Definition 2.1. If (G, φ) is a $(n+1)$-group, then the set

$$H = \{u \in G |\ \varphi(u, x, \underset{n-2}{u}, \overline{u}) = x,\ x \in G\}$$

is called the $(n+1)$-center of (G, φ) and it is denoted by $H = Z_{n+1}(G, \varphi)$.

Remark 2.1. From Theorem 2.1 it follows that if $H \neq \emptyset$ then $(Z_{n+1}(G, \varphi), \varphi)$ is a sub-$(n+1)$-group in (G, φ).

We will establish a relation between the $(n+1)$-center of a $(n+1)$-group and the centers of the reduced Hosszú groups.

Let $u \in H$ and $(G, \circ) = Red_u(G, \varphi)$ the reduced Hosszú group through u.

Theorem 2.2. *If $Z_u = Z(G, \circ)$ is the center of the group $(G, \circ) = Red_u(G, \varphi)$, then $Z_u = H$, for every $u \in H$.*

Proof. If $v \in H$, then

$$\alpha_v(x) = v \circ \alpha_u(x) \circ (\alpha_u(v))' = v \circ x \circ v'$$

So the equality $\alpha_v(x) = x$, $x \in G$ is equivalent with

$$v \circ x = x \circ v, \quad x \in G,$$

thus $v \in Z_u$. □

Remark 2.2.
- If $H \neq \emptyset$ (there exists $u \in G$ such that $\varphi(u, x, \underset{n-2}{u}, \overline{u}) = x$, $x \in G$), then all the centers of the reduced Hosszú groups through elements of H coincide (are equal to H).
- Let us denote $(G, *) = Red_g(G, \varphi)$. If $g \in G \setminus H$ then $Z_g = Z(G, *) \neq H$ ($g \in Z(G, *)$ but $g \notin H$).

Next we give a theorem of characterization of $(n+1)$-groups derivable from groups.

Let $(G, \cdot)$ be group and denote by $Int(G, \cdot)$ the set of inner automorphisms

$$Int(G, \cdot) = \{i_g : G \to G |\ i_g(x) = g \cdot x \cdot g^{-1},\ x \in G\}.$$

Theorem 2.3. *If $(G, \varphi) = Ext_{\alpha,a}(G, \cdot)$ is an $(n+1)$-group, then the following statements are equivalent:*

a) α is an inner automorphism of $(G, \cdot)$.

b) The $(n+1)$-group (G, φ) is a derived group.

c) For all $u \in G$, the reduced automorphism α_u is a inner automorphism.

Proof. a) ⇒ b) If 1 is the unit element of $(G, \cdot)$ then the reduced automorphism α_1 is $\alpha_1 = \alpha$, which is an inner automorphism, so

$$\alpha_1(x) = b \cdot x \cdot b^{-1}, \quad x \in G.$$

We have:

$$\varphi(x_1, x_2, \ldots, x_n, x_{n+1}) = x_1 \cdot \alpha(x_2) \cdot \alpha^2(x_3) \cdot \cdots \cdot \alpha^{n-1}(x_n) \cdot \alpha^n(x_{n+1}) \cdot a$$

and

$$\alpha^n(x) = a \cdot x \cdot a^{-1}.$$

It follows

$$\varphi(x_1, x_2, \ldots, x_n, x_{n+1}) = x_1 \cdot b \cdot x_2 \cdot b \cdot x_3 \cdot \cdots \cdot x_n \cdot b \cdot x_{n+1} \quad \text{and} \quad a = b^n.$$

For $v = b^{-1}$, the reduced automorphism is

$$\alpha_{b^{-1}}(x) = b^{-1} \cdot \alpha_1(x) \cdot \alpha_1(b) = b^{-1} \cdot b \cdot x \cdot b^{-1} \cdot b = x, \quad x \in G,$$

thus $\alpha_{b^{-1}} = 1_G$, then $(G, \varphi) = Der_1(G, *)$, where $x * y = x \cdot b \cdot y$, $x, y \in G$.

b) ⇒ c) If $(G, \varphi) = Der_a(G, \cdot)$ then $(G, \varphi) = Ext_{1_G,a}(G, \cdot)$, $Red_1(G, \varphi) = (G, \cdot)$, $\alpha_1 = 1_G$ and let $(G, \circ) = Red_u(G, \varphi)$ with $u \in G$.

According to the formula 1.3 we have

$$\alpha_u(x) = u \cdot \alpha_1(x) \cdot \alpha_1(u^{-1}) = u \cdot x \cdot u^{-1}, \quad x \in G$$

and therefore $\alpha_u \in Int(G, \varphi)$.

c) ⇒ a) In particular for $u = 1$ we have $\alpha_1 = \alpha$ and therefore $\alpha \in Int(G, \cdot)$. □

Remark 2.3.
- The only $(n+1)$-ary operations of $(n+1)$-group which are derivable from a group $(G,\cdot)$ have the form:

$$\varphi(x_1,x_2,\dots,x_n,x_{n+1}) = x_1\cdot b\cdot x_2\cdot\dots\cdot b\cdot x_n\cdot b\cdot x_{n+1},$$

$x_1,x_2,\dots,x_n,x_{n+1}\in G$, where $b\in G$ such that $b^n = a\in Z(G,\cdot)$.
- For the example 1.0 given by Dudek and Michalski in [2] we obtain that the only $(n+1)$-groups derivable from the group $(\mathbb{Z}_k,+)$, $k\in\mathbb{N}$, $k\geq 2$ have the $(n+1)$-ary operation of the form:

$$\varphi(x_1,x_2,\dots,x_n,x_{n+1}) = x_1+x_2+\dots+x_n+x_{n+1}+a,$$

$x_1,x_2,\dots,x_n,x_{n+1}\in\mathbb{Z}_k$, with arbitrary $a\in\mathbb{Z}_k$. The operation of example 1.0 has not this form, thus this $(n+1)$-group is not a derived $(n+1)$-group.

Supposing that the groups $(G,\cdot)$ and $(G,*)$ are fixed and we consider the $(n+1)$-group $(G,\varphi) = Ext_{\alpha,a}(G,\cdot)$ we will determine the conditions under which (G,φ) is derived from $(G,*)$, that is

$$(G,\varphi) = Der_c(G,*),\quad c\in Z(G,*).$$

For this we recall a result of [5].

Theorem 2.4. [5] *If (G,φ) and (H,ψ) are $(n+1)$-groups then the function $f: G\to H$ is a $(n+1)$-group morphism if and only if for any $u\in G$ we have:*

a) f is a morphism of their reduces groups

$$(G,\cdot) = Red_u(G,\varphi)\quad\textit{and}\quad (G,\circ) = Red_{f(u)}(H,\psi).$$

b) $f(a_u) = b_{f(u)}$, where $a_u = \varphi(\underset{n+1}{u})$ and $b_{f(u)} = \psi(\underset{n+1}{f(u)})$.

c) $f\circ\alpha_u = \beta_{f(u)}\circ f$, where $\alpha_u(x) = \varphi(u,x,\underset{n-2}{u},\overline{u})$, $x\in G$ and $\beta_{f(u)}(y) = \psi(f(u),y,\underset{n-2}{f(u)},\overline{\overline{f(u)}})$, $y\in H$, $\overline{u}$ is the skew element of u in (G,φ) and $\overline{\overline{f(u)}}$ is the skew element of $f(u)$ in (H,ψ).

Theorem 2.5. *The $(n+1)$-group $(G,\varphi) = Ext_{\alpha,a}(G,\cdot)$ is derived from the group $(G,*)$, $(G,\varphi) = Der_c(G,*)$ if and only if:*

a) $x\cdot y = x*1'*y$, for all $x,y\in G$.

b) $a = \underbrace{1*1*\dots*1}_{n+1}*c$

c) $\alpha(x) = 1*x*1'$, for all $x\in G$

where 1 is the unit element of $(G,\cdot)$ and $1'$ is the inverse of 1 in $(G,*)$.

Proof. The equality $Ext_{\alpha,a}(G,\cdot) = Der_c(G,*)$ is equivalent with the condition that $f = 1_G: G\to G$ is a morphism (isomorphism) of $(n+1)$-groups.

We have $(G,\cdot) = Red_1(G,\varphi)$ and $(G,*) = Red_e(G,\varphi)$, where e is the unit element in $(G,*)$.

The conditions a), b), c) from Theorem 2.4 [5], for $u=1$ become

a) $f(x\cdot y) = f(x)\cdot f(y) = x\cdot y = x*1'*y$, $x,y\in G$ (see 1.3)

b) $f(1\cdot\alpha(1)\cdot\dots\cdot\alpha^n(1)\cdot a) = f(1)*f(1)*\dots*f(1)*c$ or $a = 1*1*\dots*1*c$

c) $f\circ\alpha_1 = \beta_1\circ f$ or $\alpha_1 = \beta_1$.

But $\alpha_1 = \alpha$ and $\beta_1(x) = 1*x*\underbrace{1*\dots*1}_{n-2}*\overline{\overline{1}}*c$, and from $\varphi(\underset{n}{1},\overline{\overline{1}}) = 1$ follows that $1*\dots*1*\overline{\overline{1}}*c = 1'$, thus $\alpha(x) = 1*x*1'$, $x\in G$. $\square$

Corollary 2.1. *If* $(G, \varphi) = Der_a(G, \cdot)$ *and* $(G, \psi) = Der_c(G, *)$ *then* $\varphi = \psi$ *if and only if*:

a) $x \cdot y = x * 1' * y$, $x, y \in G$

b) $a = \underbrace{1 * \cdots * 1}_{n+1} * c$

c) $1 \in Z(G, *)$.

Proof. For $\alpha = 1_G$ in the Theorem 2.5 the condition c) becomes $x = 1 * x * 1'$ or $x * 1 = 1 * x$, $x \in G$, thus $1 \in Z(G, *)$. $\square$

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