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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

Computation of bounds for polynomial roots

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ABSTRACT. We discuss some old and new results on bounds for roots of univariate polynomials, emphasizing on new methods concerning real roots of univariate polynomials with real coefficients. There are presented new methods for computing bounds for positive roots of univariate polynomials. We compare our results with known upper bounds for positive roots.

1. INTRODUCTION

The computation of bounds for roots of polynomials is important for computation of roots and for estimation of polynomial sizes. Several bounds exist for the absolute values of the roots of a univariate polynomial with complex coefficients. They can be applied for obtaining bounds for real roots.

However for real roots better results can be obtained using bounds for positive roots. Until recently only the bound of Lagrange was used. We propose a general method for the computation of bounds for positive roots. The new bounds extend our theorem from 2005 (JUCS, 2005) and subsequent generalizations of Akritas-Strzeboński-Vigklas (2006, 2007) and Ştefănescu (2007). Our method is based on the choice of two families of parameters, and we discuss the problem of the optimal choice.

1.1. **Known Bounds for Positive Roots.** There exist many methods for computing bounds for roots of univariate polynomials with complex coefficients. They can be applied successfully for the case of real positive roots. All key results are based on the following

Theorem 1.1 (A.–L. Cauchy)**.** *All the roots of the nonconstant polynomial*

$$P(X) = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{C}[X]$$

are contained in the disk $|z| \le \xi$, *where* ξ *is the unique positive solution of the equation*

$$|a_n|X^n = |a_0| + |a_1|X + \cdots + |a_{n-1}|X^{n-1}. \tag{1.1}$$

Among the specific results on positive roots the most known was obtained by Lagrange.

Theorem 1.2 (Lagrange)**.** *Let* $P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$, *with all* $a_i \ge 0$, $a_0, a_{m+1} > 0$. *Let*

$$A = \max\left\{a_i\,;\, \text{coeff}\,(X^{d-i}) < 0\right\}.$$

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The number

$$1 + \left(\frac{A}{a_0}\right)^{1/(m+1)}$$

is an upper bound for the positive roots of P.

Remark 1.1. We observe that the bound of Lagrange gives bounds larger than 1. But there exist important classes of polynomials having real roots in the interval $(-1, 1)$, for example Legendre orthogonal polynomials.

For obtaining also bounds smaller than 1 the bounds of Kioustelidis [3] and that of Ştefănescu [8] give more accurate results.

Theorem 1.3 (Kioustelidis)**.** *Let* $P(X) = X^d - b_1X^{d-m_1} - \cdots - b_kX^{d-m_k} + g(X)$, *with* $g(X)$ *having positive coefficients and* $b_1 > 0, \ldots, b_k > 0$. *The number*

$$K(P) \,=\, 2 \cdot \max\{b_1^{1/m_1}, \ldots, b_k^{1/m_k}\}$$

is an upper bound for the positive roots of P.

Theorem 1.4. *Let* $P(X) \in \mathbb{R}[X]$ *and suppose that* P *has at least one sign variation. If*

$$P(X) \,=\, c_1X^{d_1} - b_1X^{m_1} + c_2X^{d_2} - b_2X^{m_2} + \cdots + c_kX^{d_k} - b_kX^{m_k} + g(X)\,,$$

with $g(X) \in \mathbb{R}_+[X]$, $c_i > 0$, $b_i > 0$, $d_i > m_i$ *for all* i, *the number*

$$S(P) = \max\left\{\left(\frac{b_1}{c_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k-m_k)}\right\}$$

is an upper bound for the positive roots of P.

Remark 1.2. For polynomials with an even number of sign changes Theorem 1.4 gives the best result. In particular the bound of Kioustelidis is double than ours. An important class of polynomials with an even number of sign changes are the classical orthogonal polynomials.

2. New Bounds for Positive Roots

We obtain new upper bounds for the positive roots of a polynomial having at least one sign change. The computation of these bounds is based on the choice of a family of parameters.

Theorem 2.1. *Let* $P(X) \,=\, aX^d - b_1X^{e_1} - b_2X^{e_2} - \cdots - b_tX^{e_t} + g(X) \in \mathbb{R}[X]$, *where* $d = \deg(P)$, $a > 0$, $b_j > 0$, $g \in \mathbb{R}_+[X]$, $e_j > e_{j+1}$ *for all* j. *An upper bound for the positive roots of* P *is given by*

$$B_1(P) \quad = \quad \max_{\substack{1\le j\le t\\ \beta_j\neq 0}} \left(\frac{b_j}{a\beta_j}\right)^{\dfrac{1}{d-e_j}}$$

for all $\beta_j \ge 0$, $\displaystyle\sum_{j=1}^{t} \beta_j = 1$.

Proof. Let $x \in \mathbb{R}$, $x > 0$. We have

$$\begin{aligned} P(x) &\geq ax^d - \sum_{j=1}^{t} b_j x^{e_j} = \left(\sum_{j=1}^{t} \beta_j\right) ax^d - \sum_{j=1}^{t} b_j x^{e_j} = \\ &= \sum_{j=1}^{t} x^{e_j} \left(a\beta_j x^{d-e_j} - b_j\right) . \end{aligned} \tag{2.2}$$

The last expression is positive if

$$x > \left(\frac{b_j}{a\beta_j}\right)^{\dfrac{1}{d-e_j}} \quad \text{for all} \quad j = 1, 2, \ldots, t\,,\ \beta_j \neq 0\,,$$

and this proves that $B_1(P)$ is an upper bound for the positive roots of P. □

Remark 2.1. We observe that Theorem 2.1 can be applied to any polynomial having positive roots.

Another bound for dominant positive roots is given by

Proposition 2.1. *Let*

$$P(X) = aX^d - b_1 X^{e_1} - b_2 X^{e_2} - \cdots - b_k X^{e_k} + g(X) \in \mathbb{R}[X]\,,$$

where $g \in \mathbb{R}_+[X]$, $a > 0$, $b_j > 0$ *for all* j, $d = \deg(P)$, $d > e_i$ *for all* i. *An upper bound for the positive roots of* P *is given by*

$$\max\left\{\left(\frac{kb_1}{a}\right)^{\dfrac{1}{d-e_1}}, \ldots, \left(\frac{kb_k}{a}\right)^{\dfrac{1}{d-e_k}}\right\}$$

Proof. We observe that for $x > 0$ we have

$$P(x) \geq \sum_{j=1}^{k} \left(\frac{a}{k} x^d - b_j x^{e_j}\right)$$

so $f(x) > 0$ provided $x^d > \frac{k}{a} x^{e_j}$ for all j. It follows that for

$$x > \max\left\{\left(\frac{kb_1}{a}\right)^{\dfrac{1}{d-e_1}}, \ldots, \left(\frac{kb_k}{a}\right)^{\dfrac{1}{d-e_k}}\right\}$$

we have $P(x) > 0$ □

We also obtain the following

Proposition 2.2. *The number*

$$\max\left\{\left(\frac{2b_1}{a_1}\right)^{\frac{1}{d_1-e_1}},\left(\frac{2^2b_2}{a_2}\right)^{\frac{1}{d_2-e_2}},\ldots,\right.$$
$$\left.\left(\frac{2^{t-1}bt_2}{a_{t-2}}\right)^{\frac{1}{d_{t-2}-e_{t-2}}},\left(\frac{2^tb_{t-1}}{a_{t-1}}\right)^{\frac{1}{d_{t-1}-e_{t-1}}},\left(\frac{2^tb_t}{a_t}\right)^{\frac{1}{d_t-e_t}}\right\}$$

is an upper bound for the positive roots of

$$P(X) = a_1X^{d_1}+a_2X^{d_2}+\cdots+a_tX^{d_t}-b_1X^{e_1}-b_2X^{e_2}-\cdots-b_tX^{e_t}+g(X)\in\mathbb{R}[X]\,.$$

Example 2.1. Let $P(X) = X^9 + X^8 - 2X^7 - 3X^5 + 1$. We obtain the following bounds for the largest positive root:

2.73	Theorem 1.2 of Lagrange
2.82	Theorem 1.3 of Kioustelidis
1.412	Theorem 1.4 of Ştefănescu

For this polynomial the largest positive root is 1.072.

For a discussion on other bounds see [2], [9] and [11].

Example 2.2. Let $P(X) = X^9 - 2X^7 - 3X^5 - X + 1$. We obtain the following bounds for the largest positive root:

2.73	Theorem 1.2 of Lagrange
2.82	Theorem 1.3 of Kioustelidis

For using Theorem 1.4 of Ştefănescu we consider the representation

$$P(X) = (\frac{1}{3}X^9 - 2X^7) + (\frac{1}{3}X^9 - 3X^5)(\frac{1}{3}X^9 - X) + 1\,.$$

We obtain the bound 2.449.

Theorem 2.1 and Propositions 2.1 and 2.2 give also the bound 2.449.

We observe that the largest positive root of the polynomial P is 1.735.

Remark 2.2. In Theorem 2.1 there are several possibilities to choose the parameters β_j. The most natural choices are given by the arrangement of the two sequences in arithmetical or geometrical progressions. For particular choices Propositions 2.1 and 2.2 give the best bounds.

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