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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} *anniversary*

Semifixed sets for multivalued φ-contractions

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ABSTRACT. The purpose of this paper is to present some semifixed set theorems for multivalued φ-contractions. Our results generalize some recent theorems by F. S. De Blasi. As an application, existence and uniqueness for the solution of a set integral equation is obtained.

1. INTRODUCTION

Let $(\mathcal{X}, d)$ be a metric space. Throughout this paper we will use the following notations and concepts:
$P(\mathcal{X}) := \{Y \subseteq \mathcal{X} | Y \neq \emptyset\}$;
$P_{cp}(\mathcal{X}) = \{Y \in P(\mathcal{X}) | Y$ is nonempty compact$\}$;
$P_{cv}(\mathcal{X}) = \{Y \in P(\mathcal{X}) | Y$ is nonempty convex$\}$;
$P_{b,cl}(\mathcal{X}) = \{Y \in P(\mathcal{X}) | \overline{Y} = Y, diam(Y) < \infty\}$;
$P_{cp,cv}(\mathcal{X}) = P_{cp}(\mathcal{X}) \cap P_{cv}(\mathcal{X})$.

(1) $D : P(\mathcal{X}) \times P(\mathcal{X}) \to \mathbb{R}_+,\ D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\}$.
D is called the gap functional between A and B.
In particular, if $x_0 \in \mathcal{X}$ then $D(x_0, B) := D(\{x_0\}, B)$.

(2) $diam : P(\mathcal{X}) \to \mathbb{R}_+ \cup \{+\infty\}, diamA := sup\{d(a, b)|a, b \in A\}$.

(3) $\rho : P(\mathcal{X}) \times P(\mathcal{X}) \to \mathbb{R}_+ \cup \{+\infty\}, \rho(A, B) = \sup\{D(a, B) \mid a \in A\}$.
ρ is called the (generalized) excess functional.

(4) $h : P(\mathcal{X}) \times P(\mathcal{X}) \to \mathbb{R}_+ \cup \{+\infty\}, h(A, B) = \max\{\rho(A, B), \rho(B, A)\}$.
h is the (generalized) Pompeiu-Hausdorff functional on $P(\mathcal{X})$ ([3]).

Let $\mathcal{A}, \mathcal{B}$ be two families of nonempty subsets of $\mathcal{X}$ and let $P(\mathcal{B})$ be the family of all nonempty subsets of $\mathcal{B}$.

Definition 1.1. Let $\phi : \mathcal{A} \to P(\mathcal{B})$ such that there exists on $F \in \phi(A)$ satisfying a relation of the type $A \subset F, A \supset F, A \cap F \neq \emptyset$, for any set $A \in \mathcal{A}$ is called a semifixed set of multivalued ϕ. Moreover, a fixed set for ϕ is any set $A \in \mathcal{A}$ satisfying $A \in \phi(A)$.

Let $\mathcal{X}$ be a Banach space and denote $\mathcal{K} := P_{cp}(P_{cp}(\mathcal{X}))$.
The space $\mathcal{K}$ is endowed with the Pompeiu-Hausdorff distance H induced by the metric h of $P_{cp}(\mathcal{X})$, i.e $H(\mathcal{A}, \mathcal{B}) := max\{e(\mathcal{A}, \mathcal{B}), e(\mathcal{B}, \mathcal{A})\}$, where $e(\mathcal{A}, \mathcal{B}) = \sup\limits_{A \in \mathcal{A}} \inf\limits_{B \in \mathcal{B}} h(A, B)$.

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For $A, A', B, B' \in P_{cp}(\mathcal{X})$ and $\lambda \in \mathbb{R}$, the set:
$D(A,B) = inf\{||a-b|| \,|a \in A, b \in B\}$, we have:

i) $D(A,B) = D(B,A)$,
ii) $D(A,B) = 0$ if only if $A \cap B \neq \emptyset$,
iii) $D(\lambda A, \lambda B) = |\lambda| D(A,B)$,
iv) $D(A,B) \leq D(A',B') + h(A,A') + h(B,B')$,
v) $h(A,B) \leq diam(A) + diam(B) + D(A,B)$.

The function D is continuous on $P_{cp}(\mathcal{X}) \times P_{cp}(\mathcal{X})$,
$|\sup\limits_{B\in\mathcal{B}} D(A,B) - \sup\limits_{B\in\mathcal{B}} D(A',B)| \leq h(A,A')$.
Define set $\Delta(\mathcal{A},\mathcal{B}) = max\{f(\mathcal{B},\mathcal{A}), f(\mathcal{A},\mathcal{B})\}$, for $\mathcal{A},\mathcal{B} \in \mathcal{K}$, where
$f(\mathcal{A},\mathcal{B}) = \inf\limits_{A\in\mathcal{A}} \sup\limits_{B\in\mathcal{B}} D(A,B)$ and $f(\mathcal{B},\mathcal{A}) = \inf\limits_{B\in\mathcal{B}} \sup\limits_{A\in\mathcal{A}} D(B,A)$.

Definition 1.2. *(*[5]*)* A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function if it satisfies:

i) φ is monotone increasing,
ii) $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0, for all $t > 0$.

Remark 1.1. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function then $\varphi(0) = 0$ and $\varphi(t) < t$, for every $t > 0$.

Example 1.1. The functions $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_1 = at$ (where $a \in]0,1[$) and
$\varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi_2(t) = \frac{t}{1+t}$ are comparison functions.

Definition 1.3. A map $\phi : \mathcal{A} \to \mathcal{K}$ is said compact if its range
$\phi(\mathcal{A}) = \{Y \in P_{cp}(\mathcal{X}) | Y \in \phi(X)$ for some $X \in \mathcal{A}\}$ is precompact in $P_{cp}(\mathcal{X})$.

As $P_{cp}(\mathcal{X})$ is complete, ϕ is compact if and only if $\phi(\mathcal{A})$ has compact closure in $P_{cp}(\mathcal{X})$.

Definition 1.4. Let $\mathcal{A}$ be a closed subset of $\mathcal{K}$. Then $\phi : \mathcal{A} \to \mathcal{K}$ is said to be a set φ-contraction if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function and

$$\Delta(\phi(X), \phi(Y)) \leq \varphi(D(X,Y)), X, Y \in \mathcal{A}.$$

The purpose of this paper is to give some semifixed set theorems for set φ-contraction. Our results extend some previous theorems given by F. S. De Blasi in [1]. As an application, existence and uniqueness for the solution of a set integral equation is obtained.

2. Main results

Our first main result is:

Theorem 2.1. *Let $\mathcal{A}$ be a closed subset of $\mathcal{K}$ and let $\phi : \mathcal{A} \to \mathcal{K}$ be a compact and upper semicontinuous multivalued, with values $\phi(X) \subset \mathcal{A}$ for every $X \in \mathcal{A}$, satisfying the following condition:*
there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

$$\Delta(\phi(X), \phi(Y)) \leq \varphi(D(X,Y)) \text{ for every } X, Y \in \mathcal{A}. \tag{2.1}$$

Then there exists $A \in \mathcal{A}$ such that:

$$A \cap F \neq \emptyset \text{ for some } F \in \phi(A). \tag{2.2}$$

Proof. Fix $X_0 \in \mathcal{A}$, and take $X_1 \in \phi(X_0)$ such that:
$D(X_1, X_0) = \inf_{Y \in \phi(X_0)} D(Y, X_0)$.
Since the function $X \to \sup_{Y \in \phi(X_0)} D(X, Y)$ is continuous on $\phi(X_1)$, a compact set, there exists $X_2 \in \phi(X_1)$ such that:

$$\sup_{Y \in \phi(X_0)} D(X_2, Y) = \inf_{X \in \phi(X_1)} \sup_{Y \in \phi(X_0)} D(X, Y) = f(\phi(X_1), \phi(X_0)), \text{ and by (2.1),}$$

$D(X_2, X_1) \leq \sup_{Y \in \phi(X_0)} D(X_2, Y) \leq \Delta(\phi(X_1), \phi(X_0)) \leq \varphi(D(X_1, X_0))$.
Similarly, as $X \to \sup_{Y \in \phi(X_1)} D(X, Y)$ is continuous on $\phi(X_2)$, for some $X_3 \in \phi(X_2)$ one has

$$\sup_{Y \in \phi(X_1)} D(X_3, Y) = \inf_{X \in \phi(X_2)} \sup_{Y \in \phi(X_1)} D(X, Y) = f(\phi(X_2), \phi(X_1)), \text{ and thus}$$

$D(X_3, X_2) \leq \sup_{Y \in \phi(X_1)} D(X_3, Y) \leq \Delta(\phi(X_0), \phi(X_1)) \leq \varphi(D(X_2, X_1))$.
By induction, one can construct a sequence $(X_n)_{n \in \mathbb{N}} \subset \mathcal{A}$, with $X_{n+1} \in \phi(X_n)$, satisfying the following relation:
$\sup_{Y \in \phi(X_{n-1})} D(X_{n+1}, Y) = \inf_{X \in \phi(X_n)} \sup_{Y \in \phi(X_{n-1})} D(X, Y) = f(\phi(X_n), \phi(X_{n-1}))$
and $D(X_{n+1}, X_n) \leq \sup_{Y \in \phi(X_n - 1)} D(X_{n+1}, Y) \leq \Delta(\phi(X_n), \phi(X_{n-1})) \leq$
$\leq \varphi(D(X_n, X_{n-1}))$.
Then we have

$$D(X_{n+1}, X_n) \leq \varphi^n(D(X_1, X_0)), n \in \mathbb{N}. \tag{2.3}$$

By the comparison function definition, we get that $\varphi^n(D(X_1, X_0))$ converges to 0, $a \to \infty$.
Since $(X_n)_{n \in \mathbb{N}} \subset \phi(\mathcal{A})$ and ϕ is compact, there exists the subsequences (X_{n_k}) and (X_{n_k+1}) and the set $A, F \in \mathcal{A}$ such that:
$\lim_{k \to +\infty} h(X_{n_k}, A) = 0 = \lim_{k \to +\infty} h(X_{n_k+1}, F)$.
Since $X_{n_k+1} \in \phi(X_{n_k}),$ for $k \in \mathbb{N}$, and by the upper semicontinuity of ϕ, it follows that $F \in \phi(A)$.
Since $D(A, F) \leq D(X_{n_k}, X_{n_k+1}) + h(X_{n_k}, A) + h(X_{n_k+1}, F)$, we have
$D(A, F) = 0$ and the proof is complete. □

Another main result is:

Theorem 2.2. *Let $\mathcal{A} \in P_{cp}(\mathcal{X})$ and $\phi : \mathcal{A} \to \mathcal{A}$ be a continuous map satisfying the following conditions:*

(a) $\phi(\mathcal{B})$ is precompact in $\mathcal{A}$ for every bounded set $\mathcal{B} \subset \mathcal{A}$;
(b) there exists $M > 0$ such that $diam(\phi(X)) \leq M$, for every $X \in \mathcal{A}$.
(c) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = t - \varphi(t)$ is strictly increasing, onto and for which the following assertion is satisfied:

$$D(\phi(X), \phi(Y)) \leq \varphi(D(X, Y)) \textit{ for all } X, Y \in \mathcal{A}. \tag{2.4}$$

Then there exists $A \in \mathcal{A}$ such that $A \cap \varphi(A) \neq \emptyset$.

Proof. Let $X_0 \in \mathcal{A}$, and let the sequence $(X_n)_{n\in\mathbb{N}}$ given by $X_n = \phi(X_{n-1})$, $n \in \mathbb{N}^*$. We will prove first that the sequence $(X_n)_{n\in\mathbb{N}}$ is bounded.
Indeed, for every $n \in \mathbb{N}$ we have:
$D(X_n, X_0) \leq D(\phi(X_n), \phi(X_0)) + h(D(\phi(X_n), X_n)) + h(\phi(X_0), X_0)$
$\leq \varphi(D(X_n, X_0)) + h(D(X_{n+1}, X_n)) + h(X_1, X_0)$
$\leq \varphi(D(X_n, X_0)) + [diam(\phi(X_n)) + D(X_{n+1}, X_n) + diam(X_n)] + [diam(\phi(X_0)) + D(X_1, X_0) + diam(X_0)]$
$\leq \varphi(D(X_n, X_0)) + 4M + D(X_{n+1}, X_n) + D(X_1, X_0)$
$\leq \varphi(D(X_n, X_0)) + 4M + \varphi^n(D(X_1, X_0)) + D(X_1, X_0)$
$D(X_n, X_0) \leq \varphi(D(X_n, X_0)) + 4M + D(X_1, X_0) + D(X_1, X_0)$
$D(X_n, X_0) - \varphi(D(X_n, X_0)) \leq 4M + 2D(X_1, X_0)$
then $\psi(D(X_n, X_0)) \leq 4M + 2D(X_1, X_0)$,
$D(X_n, X_0) \leq \psi^{-1}(4M + 2D(X_1, X_0)) < \infty$.
By (a), there exists a subsequence (X_{n_k}) converging to some set $A \in \mathcal{A}$.
Then $(\phi(X_{n_k}))$ converges to $\phi(A)$ and thus, by (2.3) we have $A \cap \phi(A) \neq \emptyset$.
The proof is complete. □

We will use the following result (see J. Matkowski and I.A.Rus [5]):

Theorem 2.3. *Let (X, d) be a complete metric space and $g : X \to X$ be a φ-contraction (i.e $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function) and*
$d(f(x), g(y)) \leq \varphi(d(x, y))$ for each $x, y \in X$.
Then g has a unique fixed point.

We will present now an application of the previous results.
Let $\mathbf{B}_r := \{X \in P_{cp,cv}(\mathbb{R}^n) | diam(X) \leq r\}$, where $r > 0$. The space $\mathbf{B}_r$ endowed with the Pompeiu-Hausdorff metric is convex and complete.
Let $F : I \times I \times \mathbf{B}_r \to \mathbf{B}_{r/2}$, $I = [a, b]$, a set $A \in \mathbf{B}_{r/2}$ and we consider the integral set equation

$$X(t) = A + \int_a^b F(t, s, X(s)) ds. \tag{2.5}$$

By a solution of (2.5) we understand a continuous function $X : I \to \mathbf{B}_r$, which satisfies (2.5) for every $t \in I$.

Theorem 2.4. *Let $F : I \times I \times \mathbf{B}_r \to \mathbf{B}_{r/2}$ be continuous and suppose there exist a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $p : I \times I \to \mathbb{R}_+$ such that:*
$h(F(t, s, X), F(t, s, Y)) \leq p(t, s)\varphi(h(X, Y))$ for every $t, s \in I$, $X, Y \in \mathbf{B}_r$, where $\sup\limits_{t\in I} \int_a^b p(t, s) \leq 1$.
Then, for each $A \in \mathbf{B}_{r/2}$ the integral equation (2.5) has a unique solution $X(\cdot, A) : I \to \mathbf{B}_r$ which depends continuously on A.

Proof. Let $\mathcal{C}(I, \mathbf{B}_r)$ be the complete space of all continuous maps $X : I \to \mathbf{B}_r$ endowed with the uniform convergence metric.
Define an operator U by the relation:

$$U(X)(t) = A + \int_a^b F(t, s, X(s)) ds, \text{ where } X \in \mathcal{C}(I, \mathbf{B}_r) \text{ and } t \in I. \tag{2.6}$$

Since $diam(U(X)(t)) \leq diam(A) + diam(\int_a^b F(t,s,X(s))ds) \leq$
$\leq \frac{r}{2} + \int_a^b diam(F(t,s,X(s)))ds \leq r$.
Hence, $U : \mathcal{C}(I, \mathbf{B}_r) \to \mathcal{C}(I, \mathbf{B}_r)$. We will show now that U is a φ-contraction.
Indeed $h(U(X)(t), U(Y)(t)) \leq h(\int_a^b F(t,s,X(s))ds, \int_a^b F(t,s,Y(s))ds) \leq$
$\leq \int_a^b h(F(t,s,X(s)), F(t,s,Y(s)))ds \leq \int_a^b p(t,s)\varphi(h(X(s),Y(s)))ds \leq$
$\leq \int_a^b p(t,s)\varphi(\max_{s \in I} h(X(s),Y(s)))ds = \int_a^b p(t,s)\varphi(||X-Y||_{\mathcal{C}(I,\mathbf{B}_r)})ds =$
$= \int_a^b p(t,s)ds\varphi(||X-Y||_{\mathcal{C}(I,\mathbf{B}_r)}) \leq \varphi(||X-Y||_{\mathcal{C}(I,\mathbf{B}_r)})$, for each $t \in I$.
Hence

$$||U(X) - U(Y)||_{\mathcal{C}(I,\mathbf{B}_r)} \leq \varphi(||X-Y||_{\mathcal{C}(I,\mathbf{B}_r)}).$$

Then the conclusion, follows by Theorem 2.3 i.e, for each $A \in \mathbf{B}_{r/2}$ the integral equation (2.5) has a solution $X(\cdot, A) : I \to \mathbf{B}_r$ which is unique and depends continuously on A. □

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