CREATIVE MATH. & INF. Online version at http://creative-mathematics.ubm.ro/
17 (2008), No. 3, 544 - 547 Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

Behavior of symmetric reaction-diffusion problems near their critical values

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ABSTRACT. We study the behavior of the Gelfand problem in its symmetric form and in the case where the critical value of the positive parameter δ does not belong in the spectrum of solutions of the corresponding steady-state problem. Using comparison techniques we obtain suitable upper and lower time-dependent solutions, which lead us to estimations of the time-dependent solution and of the blow-up time.

1. INTRODUCTION

We shall consider the symmetric case of the classical reaction-diffusion problem (Gelfand, 1963), where the critical value δ^* of the parameter does not belong in the spectrum of solutions of the problem.

We define u^* to be the solution of the problem :

$$\frac{\partial u^*}{\partial t} = \frac{\partial^2 u^*}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial u^*}{\partial r} + 2(N-2)\exp u^*,\ 0<r<1,\ t>0$$

$$\begin{aligned} \frac{\partial u^*}{\partial r}(0,t) &= u^*(1,t)=0,\ t>0 \\ u^*(r,0) &= u_0(r),\ 0<r<1 \end{aligned} \tag{1.1}$$

In the case of $N \geq 10$ the critical value $\delta^* = 2(N-2)$ does not belong in the spectrum of solutions of the steady-state problem.

It is known that $w^* = -2\,ln\,r$ is the singular steady-state solution corresponding to $\delta^* = 2(N-2)$ in the symmetric case.

We shall obtain estimations both of the time-dependent solution and the blow-up time.

2. UPPER AND LOWER SOLUTIONS

In order to obtain estimations of the time-dependent solution of the problem we are seeking for appropriate upper and lower solutions of the problem.

Theorem 2.1. *There exists an upper solution of the problem (1.1), under well-defined restrictions.*

Received: 12.09.2008. In revised form: 13.03.2009. Accepted: 22.05.2009.
2000 *Mathematics Subject Classification*. 35K57, 35K20.
Key words and phrases. *Reaction-diffusion, upper and lower solution, stationary solution, blow-up time, spectrum.*

Proof. Defining $U = -ln(c+r^2)$, for $0 \leq r \leq r_2$ and for some $c > 0$ we have:
$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial r^2} - \frac{(N-1)}{r}\frac{\partial U}{\partial r} - 2(N-2)expU = \frac{4c}{(c+r^2)^2} > 0$, for $t > 0, 0 < r < r_2$
$\frac{\partial U}{\partial r}(0,t) = 0$ and $U(r_2,t) \geq u^*(r_2,t), t > 0$
$U(r,0) \geq u_0(r), 0 \leq r \leq r_2$

So U is an upper solution of the problem for $0 < r < r_2, t > 0$, on the assumption that :

$|u_0|<K$, $u_0(r) \leq w^*(r)$ in $0<r<1$, $t>0$

with: $\frac{\partial u_0}{\partial r} > \frac{\partial w^*}{\partial r}$ or $u_0 < w^*$ for r=1, t>0. □

Then we know (Lacey & Tzanetis, 1986) that:

$$u^* \leq U < w^* \text{ for } 0 < t < T_1, 0 \leq r \leq r_2.$$

We can find a bound on the rate of approach of u^* by putting $u^* = U - \hat{u}$ as follows:

$$\frac{\partial \hat{u}}{\partial t} \geq \frac{\partial^2 \hat{u}}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial \hat{u}}{\partial r} + 2(N-2)expU[1-exp(-\hat{u})]$$

$$\geq -K_1\hat{u}^2 + 2(N-2)expU\hat{u} + \frac{\partial^2 \hat{u}}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial \hat{u}}{\partial t}$$

where

$$K_1 = \frac{1}{2}2(N-2)sup(expU) \leq \frac{N-2}{c}.$$

We define ψ as $\psi = \frac{K_2 r^a}{t+t_0}$, with $a > 0$ and K_2, t_0 positive constants to be chosen so that ψ is a lower solution for $\hat{u}$. Indeed:

$$-K_1\psi^2 + 2(N-2)expU \cdot \psi + \frac{\partial^2 \psi}{\partial r} + \frac{(N-1)}{r}\frac{\partial \psi}{\partial r} \geq \frac{\partial \psi}{\partial t} \Leftrightarrow$$

$$\Leftrightarrow r^2[\frac{2(N-2)}{r^2+c} - \frac{K_1K_2r^a - 1}{t+t_0}] \geq -a(a+N-2)$$

By choosing $K_2 = \frac{1}{K_1}$ the above relation holds as:

$$2(N-2)(t+t_0) \geq 0 \geq (r^2+c)(r^a-1).$$

If we choose t_0 such that $\frac{K_2 r^a}{t_0} \leq \hat{u}(x,0)(= U(x) - u_0(x))$, then ψ becomes a lower solution for $\hat{u}$.

Thus, we obtain the following theorem.

Theorem 2.2. *There exists a lower solution of the problem (1,1) of the form* $\psi = \frac{K_2 r^a}{t+t_0}$ *for specific values of the positive constants* a, K_2, t_0.

3. Estimations of blow-up time

After considering the rate of approach of the time-dependent solution near the critical value of the parameter, we can now estimate a lower bound of the blow-up time of the problem.

Theorem 3.3. *The rate of the blow-up time t_b in the symmetric Gelfand problem (1.1) is of the form: $t_\ell(\delta-\delta^*)^{-1/2}$, where t_ℓ is a positive constant.*

Proof. Let $u = u^* + u_1$, with $u \le U$. Then:
$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial u_1}{\partial r} + (\delta-\delta^*)expu^* + \delta(expu - expu^*)$$
$$\le (\delta-\delta^*)expu + \delta^* u_1 expU + \frac{\partial^2 u_1}{\partial r^2} + \frac{(N-1)}{r}\frac{\partial u_1}{\partial u}$$
Note that zero is a lower solution for u_1, i.e. $u_1 \ge 0$.

Let
$$\psi_1 = (\delta-\delta^*)r^{2a+1}\left(\frac{t}{(1+c)(1+a)} + 1\right)$$
then:
$$\begin{aligned}\frac{\partial \psi_1}{\partial t} &\le (\delta-\delta^*)expU + \delta\psi_1 expU + \frac{\partial^2 \psi_1}{\partial r_2} + \frac{(N-1)}{r}\frac{\partial \psi_1}{\partial r}\\ &\Leftrightarrow \frac{r^{2a+1}}{a+1} \le 1 + r^{2a-1}\left(\frac{t}{(1+c)(1+a)}+1\right)[\delta r^2 + (2a+1)(2a+2ac+N-1)]\\ &\Leftrightarrow r^2 \le (a+1)\left(\frac{t}{(1+c)(a+1)}+1\right)[(2a+1)(2a+2ac+N-1)]\end{aligned}$$

Thus, we have proved that ψ_1 is an upper solution for u_1. For $\psi - \psi_1$ no less than zero we have:
$$u \le U - \psi + \psi_1 = U - \frac{K_2 r^a}{t+t_0} + (\delta-\delta^*)r^{2a+1}\left(\frac{t}{(1+c)(1+a)}+1\right)$$
If: $\dfrac{K_2 r^a}{t+t_0} \ge (\delta-\delta^*)\left(\dfrac{t}{(1+c)(1+a)}+1\right)r^{2a+1}$ then u is no greater than $U < w^*$.
This follows if
$$K_2 \ge (\delta-\delta^*)\left(\frac{t_0}{(1+c)(1+a)}+1\right)$$
since:
$$\frac{(\delta-\delta^*)}{(1+c)(1+a)}t^2 + \left[(\delta-\delta^*)\left(1+\frac{t_0}{(1+c)(1+a)}\right)\right]t + (\delta-\delta^*)t_0 - K_2 \le 0$$
Thus we conclude that
$$0 < t \le (\delta-\delta^*)^{-1/2}\left\{\frac{(\delta-\delta^*)\left[\left(1+\frac{t_0}{(1+c)(1+a)}\right)^2(\delta-\delta^*) + \frac{4(K_2-(\delta-\delta^*)t_0)}{(1+c)(1+a)}\right]^{1/2}}{\frac{2(\delta-\delta^*)}{(1+c)(1+a)}}\right.$$
$$\left. -(\delta-\delta^*)^{1/2}[(1+c)(1+a)+t_0]\right\}$$
For $\delta \to \delta^*$ the above inequality is satisfied if $t \le t_\ell(\delta-\delta^*)^{-1/2}$ with
$$t_\ell = \{K_2(1+c)(1+a)\}^{1/2}.$$

As

$$u \leq U < w^* \text{ at } t = t_\ell(\delta - \delta^*)^{-1/2}$$

we deduce that:

$$t_b > t_\ell(\delta - \delta^*)^{-1/2}.$$

□

Hence, we have obtained a lower bound for the blow-up time in the symmetric Gelfand problem.

4. Discussion

The behavior of the Gelfand problem in the case where the critical value of the positive parameter δ does not belong in the spectrum of solutions of the corresponding steady-state problem, still remains open in the general case.

In this paper we have used the advantages of the symmetric problem in order to estimate the time-dependent solution with lower and upper solutions and to obtain a lower bound for the blow-up time.

As an open question still remains to determine an upper bound of the blow-up time.

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