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Dedicated to Professor Iulian Coroian on the occasion of his 70^{th} anniversary

A survey on cycles embeddings in Fibonacci and extended Fibonacci cubes

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ABSTRACT. The Fibonacci and extended Fibonacci cubes are two topologies used for interconnection networks in distributed systems, inspired by the Fibonacci numbers. The possibility of embedding basic interconnection topologies in Fibonacci and extended Fibonacci cubes is an important issue that defines the properties of these topologies. In this paper we give a survey on the property of the existence of a Hamiltonian cycle in Fibonacci and extended Fibonacci cubes, property which is very important, especially in the presence of faulty links when a reconfiguration of the network can be necessary.

1. INTRODUCTION

The message exchange in a distributed system depends on the communication system or on the interconnection network. An important aspect of designing a distributed system regards the design of the communication subsystem that means the design of the interconnection network. The design of the interconnection network suppose a compromise in order to achieve some objectives such as: high transfer rate, small communication delay, simplicity, scalability, fault tolerance, optimal ratio cost / performance.

An interconnection network can be modeled by a finite graph $G = (V, E)$ with V the set of vertices and E the set of edges. The vertices of the graph represents the nodes of the network, meaning the processing elements, memory modules or switches and the edges correspond to the communication links. If the communication between processors is unidirectional then the graph is a directed graph, otherwise the graph is undirected. Two processors connected by a direct (unidirectional or bidirectional) link in the network are called neighbours. The interconnection graph of the network is referred as the network topology. A good model for an interconnection network must have some properties such as: small degree, limit due to technical reasons; small diameter, means small maximum communication delay; small average distance between vertices, means small average communication delay; maximum connectivity, means optimal fault tolerance; embedding properties, means efficient simulation of other networks; modular structure, means recursive scalability.

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The most studied interconnection topology is the hypercube, or the n-cube denoted by H_n. The hypercube has good topological properties such as symmetry, small diameter and node degree. Recursive structure and efficient communication algorithms can be described for the hypercube. A drawback in the case of the hypercube is the number of nodes which is a power of 2 and limits the choice for a network interconnection with a given number of nodes. A number of interconnection topologies, called hypercube-like topologies, like star-graph, pancake-graph, cube-connected-cycles, butterfly-graph, with similar properties as the hypercube, have been proposed in the literature.

Hsu proposed and studied in [3], [4], [5] the properties of an interconnection topology called Fibonacci cube, based on the Fibonacci numbers. The Fibonacci cube of order n has f_n nodes, $n > 1$, where f_n is the n-th Fibonacci number and the nodes can be labeled with binary strings of length $n-2$ with no consecutive 1's. Two nodes are connected if their labels differ in exactly one position. It is clear that a Fibonacci cube can be seen as resulting from a hypercube after some nodes become faulty. Properties of Fibonacci cubes were also studied in [2], [6], [9], [11], [12].

Wu generalized the Fibonacci cube topology in [10] by defining the series of extended Fibonacci cubes, $(EFC_k)_{k>0}$. The extended Fibonacci cubes are defined using the same recursive relation as the Fibonacci numbers but changing the initial conditions. The topological properties and embeddings in extended Fibonacci cubes were studied in [6], [10], [13], [14].

The Fibonacci and extended Fibonacci cubes have good topological properties, are recursively decomposable and efficient communication algorithms can be described for this class of interconnection topologies. One important quality of these topologies is that they can simulate the basic topologies as arrays, cycles, meshes.

2. PRELIMINARIES

The Fibonacci Cube topology is based on the properties of Fibonacci numbers. The Fibonacci numbers are defined as $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, for $n > 1$. According to Zeckendorf's lemma, any integer number k, $0 \leq k < f_n$ can be written as a sum of Fibonacci numbers as it follows:

$$k = \sum_{i=2}^{n-1} b_i \cdot f_i, \;\; b_i \in \{0,1\} \text{ with } b_i \cdot b_{i+1} = 0, i = \overline{2, n-2}.$$

This means that to every integer number k, $0 \leq k < f_n$, we can associate a Fibonacci code $k_F = (b_{n-1} \dots b_3 b_2)_F$ according to its representation as a sum of Fibonacci numbers. This code is a binary code which has no two consecutive 1's and is called Fibonacci code. Any binary string with no two consecutive 1's is called Fibonacci string.

The Fibonacci cube can be defined as follows:

Definition 2.1. [5] *The Fibonacci cube of order $n > 1$, denoted by Γ_n, is defined as $\Gamma_n = (V_n, E_n)$ where the set of nodes is $V_n = \{0, 1, , f_n - 1\}$ and the set of edges is $E_n = \{(i,j) | H(i_F, j_F) = 1, i, j \in V_n\}$, where $H(i_F, j_F)$ is the Hamming distance between the Fibonacci codes of nodes i and j.*

The Fibonacci cube of order n has f_n nodes and there is an edge between two nodes if their Fibonacci codes differ exactly in one position. The Fibonacci cubes of order 2, 3, 4, 5 are represented in Figure 1.

Figure 1: Fibonacci cubes Γ_2 , Γ_3, Γ_4, Γ_5

Obviously, the Fibonacci cube of order n is a subgraph of the n-cube, H_n.

A recursive definition for the Fibonacci cubes has been given in [5] as follows:

Definition 2.2. [5] *The Fibonacci cube $\Gamma_n = (V_n, E_n)$ of order n, $n > 1$, is defined recursively as $V_n = 0||V_{n-1} \cup 10||V_{n-2}$, where V_{n-1} and V_{n-2} are the set of nodes of the order $n-1$, respectively $n-2$ Fibonacci cubes and $||$ denotes the concatenation of strings, and there is an edge between two nodes if their binary representations differ exactly in one position. The initial conditions are $\Gamma_2 = (\{\lambda\}, \emptyset)$ and $\Gamma_3 = (\{0,1\}, \{(0,1)\})$.*

A Fibonacci cube of order n, Γ_n, has f_n nodes and can be recursively decomposed in two Fibonacci cubes of order $n-1$ and $n-2$. The two Γ_{n-1} and Γ_{n-2} are connected by f_{n-2} edges. The Fibonacci cube Γ_n has good properties: the nodes degree is between $n/8$ and $n-2$, the diameter is $n-2$, the node and edge connectivity are between $n/8$ and $(n-2)/3$ respectively. Basic topologies such as arrays, rings, meshes, hypercubes can be embedded in Fibonacci cubes. Two thirds of the Fibonacci cubes have an odd number of nodes and as subgraphs with odd number of nodes of the hypercube, they are not Hamiltonian.

In [10] Wu introduced a series of extended Fibonacci cubes $(EFC_k)_{k>0}$, using a recursive definition. All the cubes in the series have an even number of nodes.

Definition 2.3. [10] *The series of extended Fibonacci cubes, $(EFC_k)_{k>0}$, is defined as $EFC_k(n) = (V_k(n), E_k(n))$, $n > k+1$, where $V_k(n) = 0||V_k(n-1) \cup 10||V_k(n-2)$, $n > k+3$, and two nodes are connected by an edge in $E_k(n)$ if their binary representation differ in exactly one position. The initial conditions are $V_k(k+2) = \{0,1\}^k$, $V_k(k+3) = \{0,1\}^{k+1}$, where $\{0,1\}^k$ denotes the set of binary strings of length k.*

From this definition we can see that an extended Fibonacci cube $EFC_k(n)$ can be decomposed in two extended Fibonacci cubes $EFC_k(n-1)$ and $EFC_k(n-2)$ and each node in $EFC_k(n-2)$ is connected to a node in $EFC_k(n-1)$.

The nodes of an $EFC_k(n)$ are labelled with binary strings of length $n-2$, where the first $n-k-2$ bits represent a Fibonacci code and the last k represent a binary code. The number of nodes in $EFC_k(n)$ is $2^k f_{n-k}$, where f_{n-k} is the $(n-k)$-th Fibonacci number, $n > k+1$. We can consider the Fibonacci cube Γ_n as an extended Fibonacci cube $EFC_0(n)$. The extended Fibonacci cubes $EFC_1(3)$, $EFC_1(4)$, $EFC_1(5)$, $EFC_2(4)$, $EFC_2(5)$ and $EFC_2(6)$ are given in Figure 2.

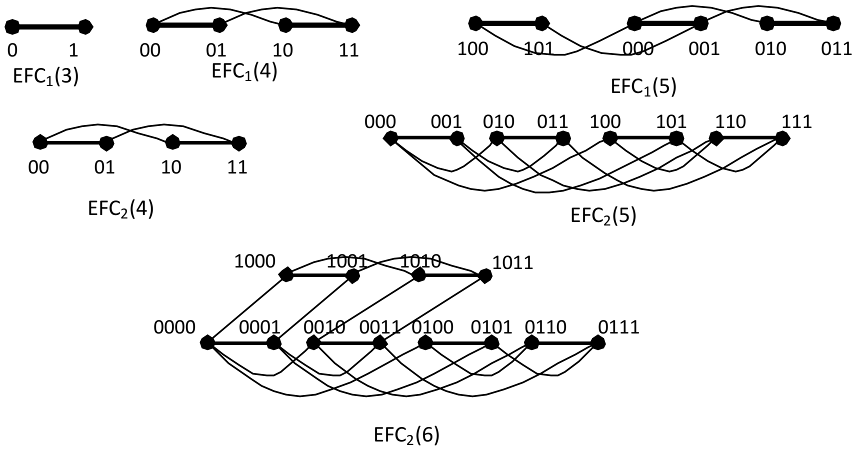

Figure 2: Extended Fibonacci cubes $EFC_1(3)$, $EFC_1(4)$, $EFC_1(5)$, $EFC_2(4)$, $EFC_2(5)$, $EFC_2(6)$

Some important properties of extended Fibonacci cubes are:

- the diameter of $EFC_k(n)$ is $n-2$ for all $k \geq 0$;
- the degree $g(u)$ of a node u in $EFC_k(n)$ is

$$\left\lceil \frac{n-(k-1)}{3} \right\rceil + (k-1) \leq g(u) \leq n-2;$$

- $EFC_{n-2}(n) = H_{n-2}$ for $n > 2$;
- $\Gamma_n = EFC_0(n) \subset EFC_1(n) \subset \ldots \subset EFC_{n-2}(n) = H_{n-2}$.

Scarano showed in [8] that, excepting the initial values, the number of nodes in $EFC_k(n)$ are distinct for different values of k and n, $k \geq 0$, $n \geq k+2$. This means that using extended Fibonacci cubes, the possibility of constructing a hypercube-like topology increases. For example, if we want to construct a cube with the number of nodes between 10 and 50 we can choose only two hypercubes, H_4 and H_5 with 16 respectively 32 nodes, but there are 11 choices for extended Fibonacci cubes: $EFC_1(6)$, $EFC_2(6)$, $EFC_0(7)$, $EFC_2(7)$, $EFC_0(8)$, $EFC_3(7)$, $EFC_1(8)$, $EFC_0(9)$, $EFC_3(8)$, $EFC_1(10)$ and $EFC_4(8)$ with the number of nodes 10, 12, 13, 20, 21, 24, 26, 34, 40, 42, 48 respectively.

3. MAIN RESULTS

A Hamiltonian path can be embedded in any Fibonacci and extended Fibonacci cube. This paths will be constructed using Gray codes for Fibonacci and extended Fibonacci cubes.

Definition 3.1. [5] *We call a Fibonacci sequence the sequence G_n recursively defined as $G_3 = \{0,1\}$, $G_4 = \{01,00,10\}$, $G_{n+2} = \{0G'_{n+1}, 10G'_n\}$, $n > 2$, where G'_k denotes the reverse sequence of the elements in G_k.*

Proposition 3.1. [5] *The Fibonacci sequence G_n defines a Gray code for the Fibonacci codes of length $n-2$, $n > 1$.*

Let $C_k = \{w_1, w_2, \ldots, w_p\}$, $p = 2^k$, be the Gray code for binary strings of length k, $H(w_i, w_{i+1}) = 1$, $i = \overline{1, p-1}$ and $G_{n-k} = \{v_1, v_2, \ldots, v_q\}$, $q = f_{n-k}$, the Gray code for Fibonacci strings of length $n-k-2$, $H(v_i, v_{i+1}) = 1$, $i = \overline{1, q-1}$.

Using these Gray codes we give the following definition for a sequence of nodes in an $EFC_k(n)$ which we show to be a Gray code for $EFC_k(n)$.

Definition 3.2. [14] *We call an extended Fibonacci sequence, the sequence $EG_k(n)$ defined as $EG_k(n) = \{v_1C_k, v_2C'_k, v_3C_k, \ldots, v_qC''_k\}$, $q = f_{n-k}$, where $v_iC_k = \{v_iw_1, v_iw_2, \ldots, v_iw_p\}$, $p = 2^k$ and $v_iC'_k = \{v_iw_p, v_iw_{p-1}, \ldots, v_iw_1\}$ represent the sequence of strings obtained from concatenation of string v_i with all the strings w_j in C_k, respectively C'_k, $i = \overline{1, q}$ and $C''_k = C_k$ if q is odd and $C''_k = C'_k$ if q is even.*

The extended Fibonacci sequence defined has $pq = f_{n-k}2^k$ elements and the strings in $EG_k(n)$ are exactly the nodes of $EFC_k(n)$.

Lemma 3.1. [14] *The extended Fibonacci sequence is a Gray code.*

With the existence of these two Gray codes for Fibonacci respectively extended Fibonacci cubes, we can give the following properties:

Proposition 3.2. [5] *There is a Hamiltonian path in any Fibonacci cube, path defined by the sequence of nodes with the codes in the Fibonacci sequence.*

Proposition 3.3. [14] *There is a Hamiltonian path in any extended Fibonacci cube, path defined by the sequence of nodes with the codes in the extended Fibonacci sequence.*

For example, for $EFC_2(6)$ the Gray code is $C_2 = \{00, 01, 11, 10\}$ and the Gray code for Fibonacci strings is $G_4 = \{01, 00, 10\}$. The extended Fibonacci sequence is $EG_2(6) = \{01C_2, 00C'_2, 10C_2\} = \{0100, 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1001, 1011, 1010\}$ and defines a Hamiltonian path in $EFC_2(6)$.

Two thirds of the Fibonacci numbers are odd numbers hence two thirds of the Fibonacci cubes have an odd number of nodes. A Fibonacci cube of order n is a subgraph of the hypercube H_n. The hypercube does not contain cycles with odd number of nodes and this means the Fibonacci cube does not contain Hamiltonian cycle. For the Fibonacci cube with even number of nodes, Γ_{3k}, $k > 1$, we can give a sequence of nodes that defines a Hamiltonian cycle.

Let $G_n = \{\alpha_1, \alpha_2, \ldots, \alpha_{f_n}\}$ be the Fibonacci sequence for the Fibonacci cube Γ_n, $n = 3k$, $k > 1$ and $G_n^0 = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_m}\}$, $G_n^1 = \{\alpha_{j_1}, \alpha_{j_2}, \ldots, \alpha_{j_l}\}$ with $i_1 < i_2 < \ldots i_m$, $j_1 < j_2 < \ldots < j_l$, $l + m = f_n$ the subsequences of G_n that have 0 respectively 1 on the last position of the codes. The nodes with the Fibonacci codes in the sequence $L = \{G_n^0, (G_n^1)'\}$, where $(G_n^1)'$ is the reverse of the sequence G_n^1, in the Fibonacci cube Γ_n, $n = 3k$, $k > 1$, defines a Hamiltonian path and the first and the last node in L are neighbours. We can give then the following result:

Theorem 3.1. [12] *The Fibonacci cubes Γ_n with an even number of nodes f_n, $n > 3$, are Hamiltonian.*

For example, in Γ_6, $G_6 = \{0100, 0101, 0001, 0000, 0010, 1010, 1000, 1001\}$ and $G_6^0 = \{0100, 0000, 0010, 1010, 1000\}$, $G_6^1 = \{0101, 0001, 1001\}$. The Hamiltonian cycle is $L = \{G_6^0, (G_6^1)'\} = \{0100, 0000, 0010, 1010, 1000, 1001, 0001, 0101\}$ and is represented in Figure 3.

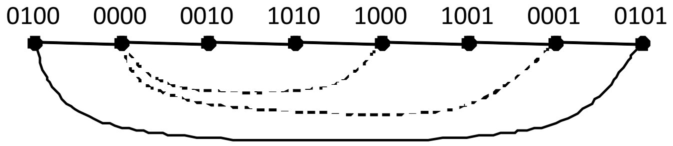

Figure 3. Hamiltonian cycle in Γ_6

Because the number of nodes in an extended Fibonacci cube is even, the extended Fibonacci cubes are all Hamiltonian. The Hamiltonian cycle in an extended Fibonacci cube can be defined in a similar way with that for Fibonacci cubes. Let $EG_k(n)$ be the extended Fibonacci sequence for $EFC_k(n)$, $EG_k(n) = \{v_1C_k, v_2C'_k, v_3C_k, \ldots, v_qC''_k\}$, $q = f_{n-k}$ with C_k and C'_k the Gray codes of order k and v_i, $i = \overline{1, f_{n-k}}$ the Fibonacci strings with length $n-k-2$. We denote $EG_k(n) = \{z_1, z_2, \ldots, z_{pq-1}, z_{pq}\}$, $pq = 2^k f_{n-k}$. We define the subsequences $EG_k^0(n) = \{z_1^0, z_2^0, \ldots, z_m^0\}$ and $EG_k^1(n) = \{z_1^1, z_2^1, \ldots, z_m^1\}$ that contain the consecutive strings in $EG_k(n)$ that have the symbol 0 respectively 1 on their last position, $m = pq/2$. The sequence $\{EG_k^0(n), (EG_k^1(n))'\}$ defines a Hamiltonian cycle in $EFC_k(n)$. We can give then

Theorem 3.2. [12] *Any extended Fibonacci $EFC_k(n)$, $k \geq 1$, $n > k+1$, is Hamiltonian.*

For $EFC_2(6)$, $EG_2(6) = \{01C_2, 00C'_2, 10C_2\} = \{0100, 0101, 0111, 0110, 0010, 0011, 0001, 0000, 1000, 1001, 1011, 1010\}$, where $C_2 = \{00, 01, 11, 10\}$ is the Gray code for binary strings of length 2. The Hamiltonian cycle in $EFC_2(6)$ is represented in Figure 4, the dashed lines are the edges not in the Hamiltonian cycle.

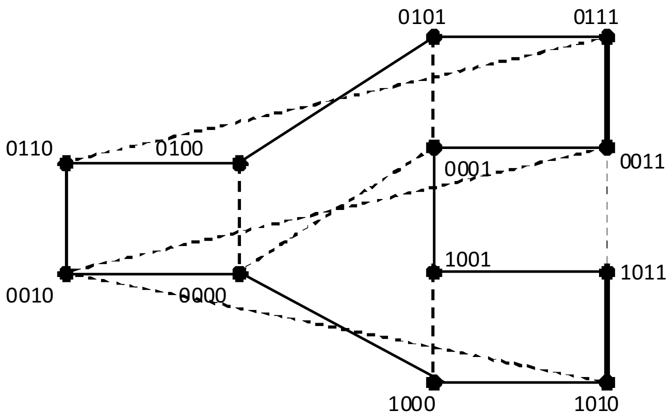

Figure: 4: Hamiltonian cycle in $EFC_2(6)$

4. CONCLUSION

The property of being Hamiltonian is an important property for the graphs which model the interconnection network of a distributed system because in the presence of faulty links, the network can be reconfigured and the communication between the nodes still can take place.

In this paper, we gave a survey on this property in Fibonacci and extended Fibonacci cubes. All these cubes contain Hamiltonian paths, and those with an even number of nodes, meaning a third of the Fibonacci cubes and all the extended Fibonacci cubes, contain Hamiltonian cycles. Much more, their topological properties are similar to those of the hypercube, so, Fibonacci and extended Fibonacci cubes are useful topologies for interconnection networks.

REFERENCES

[1] Akers, S. and Krishnamurthy, B. *A group theoretic model for symmetric interconnection networks*, IEEE Trans.Comput., 38: 555-566, 1989

[2] Cong, B., Zheng, S.Q. and Sharma, S., *On Simulations of Linear Arrays, Rings and 2D Meshes on Fibonacci Cube Networks*, Proc. 7th Intl. Parallel Processing Symp., 748-751, 1993

[3] Hsu, W.J. and Chung, M.J., *Generalized Fibonacci Cubes*, Proc. Intl. Conf Parallel Processing, 1: 299-302, 1993

[4] Hsu, W.J., *Fibonacci Cubes- Properties of an Asymmetric Interconnection Topology*, Proc. Intl. Conf. on Parralel Processing, 1249-1268, 1991

[5] Hsu, W.J., *Fibonacci Cubes- A New Interconnection Topology*, IEEE Trans. On Parallel and Distributed Systems, Vol. 4, No. 1: 3-12, 1993

[6] Karci, A., *New Interconnection Networks: Fibonacci Cube and Extended Fibonacci Based Hierarchic Networks*, Proc. 15th Intl. Conference on Information Networking, p. 869-876, 2001

[7] Katseff, H., *Incomplete hypercubes*, IEEE Trans. Comput., Vol. 37, No. 5: 604-608, 1988

[8] Scarano, V., *On the Sizes of Extended Fibonacci Cubes*, IEEE Trans. Parallel Distributed Systems, Vol. 10, 7: 764-766, 1999

[9] Stojmenovic, I., *Optimal Deadlock-free Routing and Broadcasting on Fibonacci Cube Networks*, Utilitas Mathematica 53: 159-166, 1998

[10] Wu, J., *Extended Fibonacci Cubes*, IEEE Trans. On Parallel and Distributed Systems, Vol. 8, 12: 1203-1210, 1997

[11] Zelina, I., *On embedding Fibonacci meshes on Fibonacci cube networks*, Carpathian J. Math., Vol. 22, 1-2: 173-178, 2006

[12] Zelina, I., *Hamiltonian paths and cycles in Fibonacci cubes*, Carpathian J. Math., 24(2008), No. 1-2, 149-155, 2008

[13] Zelina, I., Moldovan, G. and Pop, P.C., *Some Communication Aspects in Extended Fibonacci Cubes*, IEEE Proc. of SAINT2008, Workshop HEUNET08, Turku, Finland, publisher IEEE Computer Society Press, 245-248, 2008

[14] Zelina, I., Moldovan, G. and Taşcu, I., *On embeddings of Hamiltonian paths and cycles in Extended Fibonacci Cubes*, American Journal of Applied Sciences, Vol. 5(11), 1605-1610, 2008

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