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Remarks on some completeness conditions involved in several common fixed point theorems

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ABSTRACT.
In this note we discuss two subspace completeness conditions involved in some recent common fixed point theorems, show that they are indeed weaker than the completeness assumption of the whole ambient space and find a unifying condition for both. Using this fact, several common fixed point theorems are then reformulated under slightly more general conditions.

1. INTRODUCTION

In some very recent papers dealing with common fixed points of contractive type mappings, see [1], [5]-[9], [18] and [22]-[24], the completeness condition of the ambient space X has been replaced with conditions of the form "$g(X)$ is a complete subspace of X", or "there exists a complete metric subspace $Y \subset X$ such that $T(X) \subset Y \subset S(X)$", where g, S, T are self maps of X, under the tacit idea that such assumptions are weaker than the original one, a fact which is not quite obvious.

It is therefore the main aim of this note to show that such kind of conditions are indeed weaker than the assumption "X is a complete metric space" and then to re-state, under slightly more general conditions, some common fixed point theorems obtained in [5]-[9].

2. WEAK COMPLETENESS CONDITIONS

In this section we introduce the distinct completeness conditions and some weakly contractive conditions for the mappings of a metric space into itself.

Let (X, d) be a metric space. A subset A of X is said to be compact (or complete) in X if the closure $cl_X A$ of A in X is a compact subset (or a complete subspace) of X. If the set is compact in X, then it is complete in X, too. Consider a mapping $f : X \to X$. We put $\mathbb{N} = \{1, 2, ...\}$, and denote

$$f^0(x) = x \text{ and , } f^n(x) = f(f^{n-1}(x)), \text{ for all } x \in X \text{ and } n \in \mathbb{N}.$$

The set $P(f, x) = \{f^n(x) : n \in \mathbb{N}\}$ is called the Picard iteration or the trajectory of the point $x \in X$ relatively to f. Let

$$P(f, X) = \cup\{cl_X P(f, x) : \text{ the set } P(f, x) \text{ is complete in } X\}$$

be the set of points with the complete Picard iteration and let also

$$CP(f, X) = \cup\{cl_X P(f, x) : \text{ the set } P(f, x) \text{ is compact in } X\}$$

be the set of points with the compact Picard iteration. Obviously,

$$CP(f, X) \subseteq P(f, X), f(P(f, X)) \subseteq P(f, X) \text{ and } f(CP(f, X)) \subseteq CP(f, X).$$

Definition 2.1. A mapping $f : X \to X$ is called:

(a) with *compact range* if the set $f(X)$ is compact in X;

(b) with *complete range* if the set $f(X)$ is complete in X;

(c) with *point compact range* if the set $P(f, x)$ is compact in X for any point $x \in X$;

(d) with *point complete range* if the set $P(f, x)$ is complete in X for any point $x \in X$;

(e) with *weakly compact range* if there exists a compact subset F of X such that the set $P(f, x) \setminus F$ is finite for any point $x \in X$;

(f) a *contractive mapping* (or a contraction), if there exists a nonnegative number $k < 1$ such that $d(f(x), f(y)) \leq k \cdot d(x, y)$ for all $x, y \in X$;

(g) a *weakly contractive mapping* $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$;

(h) *a point contractive mapping* if for every point $x \in X$ there exists a nonnegative number $k = k(x) < 1$ such that $d(f^n(x), f(f^n(x))) \leq k \cdot d(f^{n-1}(x), f^n(x))$ for any $n \in \mathbb{N}$;

(i) *a weakly point contractive mapping* if $d(f(x), f(f(x))) < d(x, f(x))$ for every point $x \in X$.

Remark 2.1. Let $f : X \to X$ be a point contractive mapping with the point complete range. Then for each point $x \in X$ there exists a fixed point $x^* = lim f^n(x)$.

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Remark 2.2. If the subspace $Y \subseteq X$ is complete, then the set Y is closed in X. Hence the mapping $f : X \to X$ is with the complete range if and only if there exists a complete subspace $Y \subset X$ such that $f(X) \subset Y$. Thus we have the following implications $(a) \to (e) \to (b) \to (d)$, $(e) \to (c) \to (h) + (d) \to (c) \to (d)$, $(f) \to (g) \to (i)$ and $(f) \to (h) \to (i)$ in the Definition 2.1.

Proposition 2.1. *Let $f : X \to X$ be a weakly point contractive mapping with the point compact range. Then:*
1. *f is a point contractive mapping.*
2. *For each point $x \in X$ there exists a fixed point $x^* = lim f^n(x)$.*

Proposition 2.2. *Let $f : X \to X$ be a contractive mapping. The following assertions are equivalent.*
1. *$P(f, X) \neq \emptyset$.* 2. *$CP(f, X) \neq \emptyset$.*
3. *f is a mapping with the point compact range, i.e.,*

$$CP(f, X) = P(f, X) = X.$$

4. *There exist a unique fixed point of the mapping f.*

By the virtue of the next examples, the conditions $(a) - (i)$ from the Definition 2.1 are distinct.

Example 2.1. Let X be the space of irrational numbers. There exist a metric d on X and a mapping $f : X \to X$ such that:
(i) the metric space (X, d) is complete;
(ii) $2d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$;
(iii) $f(X)$ is an analytical not a Borel subset of X and, in particular, $f(X)$ is not a complete subspace of X.

Proof. Indeed, let $X_n = \{t \in [2^{-n}, 2^{-n+1}] : t$ is a irrational number $\}$ and $\widetilde{X} = \{0\} \cup (\cup_{n\in\mathbb{N}} X_n$. The spaces $\widetilde{X}$ and X_n are homeomorphic to the space of irrational numbers. For each $n \in \mathbb{N}$, fix a homeomorphism $h_n : X_{n+1} \to X_n$. On X_2 fix a complete metric d_2. There exists a continuous mapping $f_1 : X_1 \to X_2$ such that $F_1 = cl_X f_1(X_1)$ is a compact subset of $\widetilde{X}$.

We can consider that $Y_2 = f_1(X_1)$ is an analytical not Borel subset of X (see [21], §38, VI). On X_1 consider the complete metric

$$d_2(x, y) = d_2(h_1^{-1})(x), h_1^{-1})(y)) + 2d_2(f_1(x), f_1(y)), \text{ for all } x, y \in X_1.$$

We assume that $d_2(x, y) \leq 2^{-1}$ for all $x, y \in X_2$. By induction, we construct the complete metric $d_{n+1}(x, y) = 2^{-1}d_n(h_n(x), h_n(y))$ for all $x, y \in X_{n+1}$ and $n \geq 2$. On X consider the complete metric d, where:
1) $d(x, y) = d_n(x, y)$ for all $x, y \in X_n$ and $n \in \mathbb{N}$;
2) if $n, m \in$, $m < n$, $x \in X_n$ and $y \in X_m$, then

$$d(x, y) = d(y, x) = \Sigma\{2^{-i} : m - 1 \leq i \leq n\};$$

3) if $n \in$ and $x \in X_n$, then $d(x, 0) = d(0, x) = \Sigma\{2^{-i} : n - 1 \leq i, i \in \mathbb{N}\}$.
Now we consider the mapping $f : X \to X$, where $f(0) = 0$, $f_1 = f|X_1$ and $f(x) = h_n^{-1}(x)$ for all $x \in X_n$ and $n \geq 2$. By construction

$$d(f(x), f(y)) \leq 2^{-1}d(x, y) \text{ for all } x, y \in X.$$

Denote $F_{n+1} = f^n(F_1)$ for any $n \in \mathbb{N}$, $F = \{0\} \cup (\cup\{F_n : n \in \mathbb{N}\})$, $Y = X_1 \cup F$ and $g = f|Y$. Then:
1) (Y, d) is a complete metric space;
2) $2d(g(x), g(y)) < d(x, y)$ for all $x, y \in Y$;
3) $g(Y)$ is an analytical not Borel subset of Y and, in particular, $g(Y)$ is not a complete subspace of Y;
4) $g(Y) \subseteq F$ and F is a compact subset of Y. Now we put

$$Z = Y \times X, \rho((x, u), (y, v)) = d(x, y) + d(u, v) \text{ for all } (x, u), (y, v) \in Z, \Phi = F \times \{0\}$$

and $\varphi(x, u) = (g(x), 0) = (f(x), 0)$ for all $(x, u) \in Z$. Then:
1) (Z, ρ) is a complete metric space homeomorphic to the space of irrational numbers;
2) $2\rho(\varphi(x), \varphi(y)) < \rho(x, y)$ for all $x, y \in Z$;
3) $\varphi(Z)$ is an analytical not Borel subset of Z and, in particular, $\varphi(Z)$ is not a complete subspace of Z;
4) $\varphi(Z) \subseteq \Phi$ and Φ is a compact subset of Z. □

Example 2.2. Let $X_0 = \{b\}$, (X_1, d_1) be a metric space and (X_2, d_2) be a metric space for which:
a) $b \notin X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$;
b) $d_1(x, y) \leq 1$ for all $x, y \in X_1$;
c) $d_2(x, y) \geq 2$ for all distinct points $x, y \in X_2$.

We put $X = \{b\} \cup X_1 \cup X_2$, $d(x, y) = d_1(x, y)$ for all $x, y \in X_1$,

$$d(x, y) = d_2(x, y) \text{ for all } x, y \in X_2,\ d(x, y) = d(y, x) = 1 \text{ for } x \in X_1,\ y \in X_2,$$

$$d(x, 0) = d(0, x) = 2^{-1} \text{ and } , d(y, 0) = d(0, y) = 1 \text{ for x } \in X_1 \text{ and } y \in X_2.$$

Fix some mapping $f : X \to X$, where $f(X_2) \subseteq X_1$ and $f(X_1) = \{b\}$.
By construction, the mapping f has the following properties:

1) $2d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$, i.e. f is a contractive mapping;
2) b is the unique fixed point of the mapping f;
3) f is a mapping with the point compact range and the Picard iteration $P(f, x)$ is finite for any point $x \in X$;
4) if the metric space X_1 is complete, then the space X is complete too;
5) if the metric space X_1 is locally compact, then the space X is locally compact too;
6) f is a mapping with the weakly compact range $\{b\}$;
7) if the set $cl_X f(X_2)$ is not compact, then f is a mapping without the compact range;
8) if X_1 is a non-analytic subspace of the segment $[0, 1]$ of the space of reals $\mathbb{R}$ and $X_1 = f(X_2)$, then f is a mapping without the complete range;
9) if the set X_2 is uncountable, then the space X is not separable.

Example 2.3. Let X be the space of rational numbers, $d(x, y) = |x - y|$ and $f(x) = 2^{-1}x$. The metric space (X, d) is not complete, f is a contractive mapping with a point compact range and with a unique fixed point 0. The mapping f is not with a weakly compact range.

Example 2.4. Let $A = \{(1 + 2^{-n}, 2^{-n}) : n \in \mathbb{N}\}$, $A = \{(-1 - 2^{-n}, 2^{-n}) : n \in \mathbb{N}\}$, $X = \{(1, 0)\} \cup A \cup B$ and (X, d) be a subspace of the Euclidean plane. The space (X, d) is not complete and admit an equivalent complete metric.

Proof. We put $f(1, 0) = (1, 0)$ and $f(1 + 2^{-n}, 2^{-n}) = (1 + 2^{-n-1}, 2^{-n-1})$, $f(-1 - 2^{-n}, 2^{-n}) = (-1 - 2^{-n-1}, 2^{-n-1})$ for each $n \in \mathbb{N}$.
The mapping f is weakly contractive and not contractive, $CP(f, X) = P(f, X) = A \cup \{(1, 0)\}$ and $(1, 0)$ is the unique fixed point of the mapping f.

If $C = \{(1, 0)\} \cup A$, then C is a compact subspace, $f(A) \subseteq A$, $f(B) \subseteq B$, $f(C) \subseteq C$ and the mappings $g = f|C : C \to C$, $h = f|B : B \to B$ are contractive.

Now let $Y = X \cup \{-1, 0)\}$, $\varphi(-1, 0) = (-1, 0)$ and $f = \varphi|X$. Then:
1) the mapping φ is not weakly contractive;
2) φ is a point contractive mapping;
3) Y is a compact space. □

Remark 2.3. In view of Bessaga's converse of Banach contraction principle [11], see also [27]-[30], let $f : X \to X$ be a self mapping of an abstract set X such that the set of fixed points of f, $Fix\,(f)$, is nonempty and let k be such that $0 < k < 1$. Fix a point $b \in Fix\,(f)$. Let Y be the set of all points $x \neq b$ in X for which the Picard iteration $P(f, x)$ is finite and let $Z = X \setminus Y$. Then by virtue of Bessaga's theorem, there exists a complete metric on Z such

$$d(f(x), f(y)) \leq k\, d(x, y), \text{ for all } x, y \in Z.$$

If $X = Z$, then d is a complete metric on X and f is a contraction. Assume $X \neq Z$. Fix a point $c \in Y$. We extend the metric d on X in the following way:

$$d(x, y) = 1,\ x, y \in Y;\ d(y, z) = d(z, y) = d(y, c) + 1 + d(b, z),\ \ y \in Y,\ z \in Z.$$

The metric d is complete on X and relatively to the metric d the mapping f is:
1) with point compact range;
2) point contractive at each point $x \in Z \cup Fix\,(f)$.

As a conclusion of this section, we can use the general completeness condition *"the closure $\overline{S(X)}$ of the set $S(X)$ in X is a complete subspace of X"*, which obviously includes the two particular cases already mentioned:
a) the metric space (X, d) is complete; b) $S(X)$ is a complete subspace of X.

In several common fixed point theorems that involve two mappings T and S, see [22]-[23], the following alternative completeness condition have been also considered: there exists a complete metric subspace $Y \subset X$ such that $T(X) \subset Y \subset S(X)$, which implies the inclusion $T(X) \subset S(X)$, explicitly assumed in most papers on this topic, see [1], [5]-[10], [15]-[17], [22]-[26].

3. Some common fixed point theorems under weak completeness conditions

The next results extend Theorems 3 and 4 in [8], and Theorem 4 in [10], respectively by weakening the completeness condition in two different ways.

Theorem 3.1. *Let (X, d) be a metric space and let T, $S : X \to X$ be two mappings for which there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta \cdot d(Sx, Sy) + Ld(Sy, Tx)\,, \quad \textit{for all}\ \ x, y \in X\,. \tag{3.1}$$

Assume $T(X) \subset S(X)$ and that the closure $\overline{S(X)}$ of the set $S(X)$ in X is a complete subspace of X. Then T and S have a coincidence point in X.

Moreover, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by (3.3) converges to some coincidence point x^* of T and S, with the following error estimate

$$d(Sx_{n+i-1}, x^*) \leq \frac{\delta^i}{1-\delta}\, d(Sx_n, Sx_{n-1})\,, \quad n = 0, 1, 2, \ldots;\ i = 1, 2, \ldots \tag{3.2}$$

Proof. Let x_0 be an arbitrary point in X. Since $T(X) \subset S(X)$, we can choose a point x_1 in X such that $Tx_0 = Sx_1$ Continuing in this way, for a x_n in X, we can find $x_{n+1} \in X$ such that

$$Sx_{n+1} = Tx_n, \; n = 0, 1, \dots \tag{3.3}$$

If $x := x_n, \; y := x_{n-1}$ are two successive terms of the sequence defined by (3.3), then by (3.1) we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq L \cdot d(Sx_n, Tx_{n-1}) + \delta \cdot d(Sx_{n-1}, Sx_n),$$

which, in view of the fact that (3.3) implies $d(Sx_n, Tx_{n-1}) = 0$, yields

$$d(Sx_{n+1}, Sx_n) \leq \delta \cdot d(Sx_n, Sx_{n-1}), \; n = 0, 1, 2 \dots. \tag{3.4}$$

Now by induction, from (3.4) we obtain

$$d(Sx_{n+k}, Sx_{n+k-1}) \leq \delta^k \cdot d(Sx_n, Sx_{n-1}), \; n, k = 0, 1, \dots \; (k \neq 0), \tag{3.5}$$

and then, for $p > i$, we get after straightforward calculations

$$d(Sx_{n+p}, Sx_{n+i-1}) \leq \frac{\delta^i(1-\delta^{p-i+1})}{1-\delta} \cdot d(Sx_n, Sx_{n-1}), \; n \geq 0; \; i \geq 1. \tag{3.6}$$

Take $i = 1$ (3.6) and the, by an inductive process, we get

$$d(Sx_{n+p}, Sx_n) \leq \frac{\delta}{1-\delta} \cdot d(Sx_n, Sx_{n-1}) \leq \frac{\delta^n}{1-\delta} \cdot d(Sx_1, Sx_0), \; n = 0, 1, 2 \dots,$$

which shows that $\{Sx_n\}$ is a Cauchy sequence.

Since $\overline{S(X)}$ is a complete subspace of X, there exists a x^* in $\overline{S(X)}$ such that

$$\lim_{n\to\infty} Sx_{n+1} = x^*. \tag{3.7}$$

We can find $p \in X$ such that $Sp = x^*$. By (3.3) and (3.4) we further have

$$d(Sx_n, Tp) \leq \delta d(Sx_{n-1}, Sp) \leq \delta^{n-1} d(Sx_1, Sp),$$

which shows that we also have

$$\lim_{n\to\infty} Sx_n = Tp. \tag{3.8}$$

Now by (3.7) and (3.8) it results now that $Tp = Sp$, that is, p is a coincidence point of T and S (or x^* is a point of coincidence of T and S). The estimate (3.2) is obtained from (3.6) by letting $p \to \infty$. □

Remark 3.4. Note that the coincidence point ensured by Theorem 3.1 is not generally unique, see Example 1 in [10].

In order to derive a common fixed point theorem from the coincidence Theorem 3.1, we have two possibilities: 1) to ensure the uniqueness of the coincidence point; or 2) to impose an additional property for the pair (S, T).

The next result adapts Theorem 2 in [10] to the case of metric spaces, by using a different weak completeness condition.

Theorem 3.2. *Let (X, d) be a metric space and let $T, S : X \to X$ be two mappings satisfying (3.1), for which there exist a constant $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$d(Tx, Ty) \leq \theta \cdot d(Sx, Sy) + L_1 d(Sx, Tx), \quad \text{for all } \; x, y \in X. \tag{3.9}$$

Assume there exists a complete metric subspace $Y \subset X$ such that $T(X) \subset Y \subset S(X)$. Then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

In both cases, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by (3.3) converges to the unique common fixed point (coincidence point) x^* of S and T, with the error estimate (3.2)

The convergence rate of the iteration $\{Sx_n\}$ is given by

$$d(Sx_n, x^*) \leq \theta \cdot d(Sx_{n-1}, x^*), \quad n = 1, 2, \dots \tag{3.10}$$

Proof. By the proof of Theorem 3.1, we have that T and S have at least a point of coincidence. Now let us show that T and S actually have a unique point of coincidence. Assume there exists $q \in X$ such that $Tq = Sq$. Then, by (3.9) we get

$$d(Sq, Sp) = d(Tq, Tp) \leq 2\delta d(Sq, Tq) + \delta d(Sq, Tp) = \delta d(Sq, Sp)$$

which shows that $Sq = Sp = x^*$, that is T and S have a unique point of coincidence, x^*.

Now if T and S are weakly compatible, by Proposition 1 it follows that x^* is their unique common fixed point. □

A stronger but simpler contractive condition that ensures the uniqueness of the coincidence point and which unifies (3.1) and (3.9), has been recently obtained by Babu et al. [2], see the general common fixed point theorem (Theorem 4) in [10].

Theorem 3.3. *Let (X,d) be a metric space and let $T,\ S: X \to X$ be two mappings for which there exist the constants $\delta \in (0,1)$ and $L \geq 0$ such that*

$$d(Tx,Ty) \leq \delta \cdot d(Sx,Sy) + L\min\{d(Sx,Tx),\ d(Sy,Ty),\ d(Sx,Ty),\ d(Sy,Tx)\}, \quad \text{for all } x,y \in X. \tag{3.11}$$

If the range of S contains the range of T and the closure $\overline{S(X)}$ of the set $S(X)$ in X is a complete subspace of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X. In both cases, for any $x_0 \in X$, the iteration $\{Sx_n\}$ defined by (3.3) converges to the unique common fixed point (coincidence point) x^* of S and T.

Proof. If $x := x_n,\ y := x_{n-1}$ are two successive terms of the sequence defined by (3.3), then by (3.11) we have

$$d(Sx_n, Sx_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \delta \cdot d(Sx_{n-1}, Sx_n) + L \cdot M,$$

where

$$M = \min\{d(Sx_n,Tx_n), d(Sx_{n-1},Tx_{n-1}), d(Sx_n,Tx_{n-1}), d(Sx_{n-1},Tx_n)\} = 0,$$

since $d(Sx_n, Tx_{n-1}) = 0$. □

Example 3.5. Let $X = [0,1]$ with the usual norm and consider $T,S : X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{x}{4}, & 0 \leq x < \dfrac{2}{3} \\ \dfrac{2}{3}, & \dfrac{2}{3} \leq x \leq 1 \end{cases}$$

and

$$Sx = \begin{cases} x, & 0 \leq x \leq \dfrac{2}{3} \\ 1, & \dfrac{2}{3} < x \leq 1, \end{cases}$$

respectively.

We have $T(X) = [0,2/3] \subset [0,2/3] \cup \{1\} = S(X)$. To show that S and T fulfill the assumptions of Theorem 3.1 we consider four cases.

Case 1. $x,y \in [0,2/3)$. In this case (3.1) reduces to the inequality

$$\left|\frac{x}{4} - \frac{y}{4}\right| \leq \delta\,|x-y| + L \cdot \left|y - \frac{x}{4}\right|,$$

which holds for all $x,y \in [0,2/3)$ and any constant $L \geq 0$ if we simply take δ such that $1 > \delta \geq 1/4$.

Case 2. $x \in [0,2/3),\ y \in (2/3,1]$. As $Tx = \dfrac{x}{4}$, $Ty = \dfrac{2}{3}$ and $Sx = x$, $Sy = 1$, condition (3.1) reduces to show that there exist the constants δ and L, $0 \leq \delta < 1$, and $L \geq 0$, such that

$$\left|\frac{x}{4} - \frac{2}{3}\right| \leq \delta\,|x-1| + L \cdot \left|1 - \frac{x}{4}\right|,\ \forall x \in [0,\frac{2}{3}). \tag{3.12}$$

As, for $x \in [0,\frac{2}{3})$ we have $\left|\frac{x}{4} - \frac{2}{3}\right| \in \left(\frac{1}{2},\frac{2}{3}\right]$ and $\left|1 - \frac{x}{4}\right| \in \left(\frac{5}{6},1\right]$, in order to have (3.12) fulfilled, it suffices to take $L \geq \dfrac{4}{5}$ and allow $0 \leq \delta < 1$ be arbitrary.

Case 3. $x \in [0,2/3),\ y = 2/3$. In this case, (3.1) reduces to show that there exist the constants δ and L, $0 \leq \delta < 1$, and $L \geq 0$ such that

$$\left|\frac{x}{4} - \frac{2}{3}\right| \leq \delta\left|x - \frac{2}{3}\right| + L \cdot \left|1 - \frac{x}{4}\right|,\ \forall x \in \left[0,\frac{2}{3}\right), \tag{3.13}$$

which, by the previous case, is indeed satisfied for any $0 \leq \delta < 1$ if we similarly take $L \geq \dfrac{4}{5}$.

Case 4. $x,y \in [2/3,1]$. In this case (3.1) holds for any constants δ and L satisfying $0 \leq \delta < 1$ and $L \geq 0$, since its left hand side is always equal to 0.

By summarizing, we conclude that S and T satisfy the contractive condition (3.1) in Theorem 3.1 with $\delta = \dfrac{1}{4}$ and $L = \dfrac{4}{5}$.

Hence, Theorem 3.1 applies and T and S have two common fixed points, namely 0 and $2/3$.

Remark. Note that T and S in Example 3.5 do not satisfy neither conditions (3.9) in Theorem 3.2 and (3.11) in Theorem 3.3, nor the contractive conditions in [1] and other related papers.

Indeed, for $x = 0$ and $y = \dfrac{2}{3}$, condition (3.9) would require that there exist the constants θ and L_1, with $0 < \theta < 1$ and $L_1 \geq 0$ such that:

$$\left|0 - \frac{2}{3}\right| \leq \theta \left|0 - \frac{2}{3}\right| + L_1 \left|0 - 0\right|,$$

which yields the contradiction $\theta \geq 1$. Thus Theorems 3.2 and 3.3 do not apply to the mappings in Example 3.5.

T and S in Example 3.5 do not satisfy Kannan's contractive condition [20], either. Indeed, for $x = 0$ and $y = \dfrac{2}{3}$ this condition would require the existence of a constant b, $0 \leq b < 1/2$, such that

$$\left|0 - \frac{2}{3}\right| \leq b \left[\left|0 - 0\right| + \left|1 - \frac{2}{3}\right|\right],$$

which obviously yields the contradiction $2 \leq b < 1/2$. Thus Theorem 2.3 in [1] do not apply to the mappings in Example 3.5.

Moreover, T and S in Example 2.1 do not satisfy Chatterjea's contractive condition in [12]. Indeed, for $x = \dfrac{2}{3} - \varepsilon$, $\varepsilon > 0$ and $y = \dfrac{2}{3}$ this condition would require the existence of a constant c, $0 \leq c < 1/2$, such that

$$\left|\frac{\frac{2}{3} - \varepsilon}{4} - \frac{2}{3}\right| \leq c \left[\left|\frac{2}{3} - \varepsilon - \frac{2}{3}\right| + \left|\frac{2}{3} - \frac{\frac{2}{3} - \varepsilon}{4}\right|\right],$$

which by letting $\varepsilon \to 0$ yields the contradiction $1 \leq c < 1/2$. Thus Theorem 2.4 in [1] do not apply to Example 3.5.

Note that in Example 3.5 all the three completeness conditions considered in Section 2 are satisfied: (X, d) is a complete metric space; $S(X)$ is a complete subspace of X; and there exists a complete metric subspace $Y = [0, 2/3] \subset X$ or $Y = [0, 2/3] \cup \{1\} \subset X$ such that $T(X) \subset Y \subset S(X)$.

Note also that, in relation to Definition 2.1, the mapping T in Example 3.5 has the following properties:

a) is with compact and complete range;

b) is with the point compact range, see the main results in [3];

c) is with the point complete range;

d) is not a contraction;

e) is not weakly contractive;

f) is weakly point contractive, see [3].

To prove d) just take $x = 1/2$, $y = 2/3$ to get $|1/8 - 2/3| \leq |1/2 - 2/3| \Leftrightarrow 4k \geq 13$ that contradicts $k < 1$.

Similarly, to prove e) take the same values $x = 1/2$, $y = 2/3$ to get the contradiction $13 < 4$.

To prove f), observe that the weakly point contractive condition (i) in Definition 2.1 may be obtained by (3.1) if we take $S = id_X$ and $y = Tx$.

Example 3.5 partially illustrates how general the common fixed point result given by Theorem 3.1 is.

To show more on its power let us also mention some common fixed point theorems which are particular cases of Theorem 3.1 and which cannot be applied to the mappings S and T in Example 3.5: Theorem 3.2, Theorem 3.3, Theorem 3.4 in [6]; Theorem 3, Theorem 4, and Corollaries 1-3 in [10]; Theorem 2, Theorem 3, and Corollaries 1-2 in [9]; Theorem 3, Theorem 4, Theorem 5, and Corollaries 1-3 in [8].

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