CREATIVE MATH. & INF.
19 (2010), No. 1, 11 - 14

Online version available at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

# Canonical connections on $k$-symplectic manifolds under reduction

ADARA M. BLAGA

ABSTRACT.
The construction of a canonical connection on a $k$-symplectic manifold is reviewed and sufficient conditions such that it should be preserved by performing Marsden–Weinstein reduction are given.

## 1. INTRODUCTION

An important result in geometric mechanics is the symplectic reduction initiated by Liouville, Arnold and Smale. It provides new symplectic manifolds when we have symmetries on the initial manifold. In a similar way, under certain assumptions [3], one can reduce a *$k$-symplectic manifold* and obtain also a $k$-symplectic manifold.

In the present paper, we shall recall the way we constructed a canonical connection on a $k$-symplectic manifold [4] and provide sufficient conditions such that this connection to give by reduction the canonical connection on the reduced manifold.

## 2. CANONICAL CONNECTION ON A $k$-SYMPLECTIC MANIFOLD

According to [1], [6],

**Definition 2.1.** A $k$-symplectic manifold is a smooth manifold $M$ together with $k$ closed 2-forms $\omega_1, \ldots, \omega_k$ and an $nk$-dimensional foliation $\mathcal{F}$ on $M$ with the following properties:

(1) $\bigcap_{\alpha=1}^{k} \ker \omega_\alpha = \{0\}$,
(2) $\omega_\alpha(X, X') = 0$, $(\forall)\ X, X' \in \Gamma(T\mathcal{F}), \alpha \in \{1, \ldots, k\}$,

where $\ker \omega_x := \{v \in T_xM \mid \omega_x(v, w) = 0,\ (\forall)\ w \in T_xM\}$.

From the definition follows that $\dim(M) = n(k+1)$. Denote by $L$ the tangent bundle of the foliation $\mathcal{F}$ and define

$$L_\alpha := \bigcap_{\beta \neq \alpha} \ker \omega_\beta,\ \ (\forall)\ 1 \leq \alpha \leq k. \tag{2.1}$$

It follows that [2]:

$(a)$ the distribution $L_\alpha$ is integrable (we denote by $\mathcal{F}_\alpha$ the foliation integral to $L_\alpha$), $\alpha \in \{1, \ldots, k\}$;
$(b)$ $L = L_1 \oplus \cdots \oplus L_k$;
$(c)$ the map $i_\alpha : L_\alpha \longrightarrow Q^*$, $X \mapsto i_X\omega_\alpha$, is an isomorphism, $\alpha \in \{1, \ldots, k\}$, where $Q$ denotes the normal bundle of $\mathcal{F}$.

**Theorem 2.1.** [4] *Let $(M, \omega_\alpha, \mathcal{F})_{1 \leq \alpha \leq k}$ be a $k$-symplectic manifold and let $Q$ be an $n$-dimensional integrable distribution on $M$ supplementary to $T\mathcal{F}$ verifying the conditions:*

$(i)$ $\omega_\alpha(Y, Y') = 0$, $(\forall)\ Y, Y' \in \Gamma(Q), \alpha \in \{1, \ldots, k\}$,
$(ii)$ $[X, Y] \in \Gamma(L_\alpha \oplus Q)$, $(\forall)\ X \in \Gamma(L_\alpha), Y \in \Gamma(Q)$,

*and such that $(i_1^*)^{-1}(\psi_1^{YY'}) = \cdots = (i_k^*)^{-1}(\psi_k^{YY'})$, for any $Y, Y' \in \Gamma(Q)$, where $\psi_\alpha^{YY'} \in L_\alpha^*$ are defined $\psi_\alpha^{YY'} : V \mapsto (\mathcal{L}_Y i_{Y'}\omega_\alpha)(V) = Y(\omega_\alpha(Y', V)) - \omega_\alpha(Y', [Y, V])$, for any $V \in \Gamma(TM)$ and for any $Y, Y' \in \Gamma(Q)$ and $\alpha \in \{1, \ldots, k\}$.*

*Then there exists a unique connection $\nabla$ on $M$ satisfying the following properties:*

(1) $\nabla\mathcal{F}_\alpha \subset \mathcal{F}_\alpha$, $(\forall)\ \alpha \in \{1, \ldots, k\}$, *and* $\nabla Q \subset Q$,
(2) $\nabla\omega_1 = \cdots = \nabla\omega_k = 0$,
(3) $\nabla_X Y - \nabla_Y X = [X, Y]$, $(\forall)\ X \in \Gamma(L), Y \in \Gamma(Q)$.

According to the decomposition $TM = L_1 \oplus \cdots \oplus L_k \oplus Q$, we defined [4] a connection $\nabla^{L_\alpha}$ on each subbundle $L_\alpha$, a connection $\nabla^Q$ on $Q$ and then we took the sum of these connections for defining a global connection on $M$:

$$\nabla_V W := \nabla_V^{L_1} W_{L_1} + \cdots + \nabla_V^{L_k} W_{L_k} + \nabla_V^Q W_Q, \tag{2.2}$$

for $V, W \in \Gamma(TM)$.

Received: 15.05.2008; In revised form: 01.12.2009; Accepted: 08.02.2010.
2000 *Mathematics Subject Classification.* 53C15, 53D05.
Key words and phrases. *$k$-symplectic manifold, connection, Marsden–Weinstein reduction.*

## 3. Connections under reduction

We shall identify the $k$-tangent bundle $T^1_kM$ with the Whitney sum of $k$ copies of $TM$ and consider the bundle morphism

$$\Omega^{\#}: T^1_kM \longrightarrow T^*M,\ \Omega^{\#}(X_1,\ldots,X_k) := \sum_{\alpha=1}^{k} i_{X_\alpha}\omega_\alpha. \tag{3.3}$$

**Definition 3.2.** [5] A $k$-Hamiltonian system is an ordered $k$-tuple of vector fields $(X_1,\ldots,X_k) \in T^1_kM$ such that there exists a smooth function $H: M \longrightarrow \mathbb{R}$, called the Hamiltonian of $(X_1,\ldots,X_k)$, with the property

$$\Omega^{\#}(X_1,\ldots,X_k) = dH. \tag{3.4}$$

We will denote by $((X_1)_H,\ldots,(X_k)_H)$ a $k$-Hamiltonian system with the Hamiltonian $H$.

**Definition 3.3.** [5] A $k$-symplectic action of a Lie group $G$ on the $k$-symplectic manifold $(M,\omega_\alpha,\mathcal{F})_{1\le\alpha\le k}$ is an action $\Phi: G\times M \longrightarrow M$ such that

$$(\Phi_g)^*\omega_\alpha = \omega_\alpha,\ \ (\forall)\, g\in G,\ \ (\forall)\, 1\le\alpha\le k, \tag{3.5}$$

where $\Phi_g: M \longrightarrow M, \Phi_g(x) := \Phi(g,x)$, and $(\xi_\alpha)_M$ are the fundamental vector fields on $M$ corresponding to the elements $\xi_\alpha \in \mathcal{G}$, $\alpha \in \{1,\ldots,k\}$.

Let $\mathcal{G}^k = \mathcal{G}\times \overset{k}{\ldots} \times\mathcal{G}$ and $\mathcal{G}^{*^k} = \mathcal{G}^*\times \overset{k}{\ldots} \times\mathcal{G}^*$, where $\mathcal{G}^*$ is the dual of the Lie algebra $\mathcal{G}$ of $G$. For $g\in G$, define ${Ad_g}^k: \mathcal{G}^k \longrightarrow \mathcal{G}^k, {Ad_g}^k(\xi_1,\ldots,\xi_k) := (Ad_g\xi_1,\ldots,Ad_g\xi_k)$, where $Ad: G\longrightarrow Aut(G)$ denotes the adjoint representation and $Ad_g = Ad(g)$, and $Ad^{*^k}_g: \mathcal{G}^{*^k} \longrightarrow \mathcal{G}^{*^k}, Ad^{*^k}_g(\mu) = \mu\circ {Ad_g}^k$.

**Definition 3.4.** [5] A momentum map for the $k$-symplectic action $\Phi: G\times M\longrightarrow M$ is a map $J: M\longrightarrow \mathcal{G}^{*^k}$ defined by

$$(X_\alpha)_{\widehat{J}(\xi_1,\ldots,\xi_k)} := (\xi_\alpha)_M,\ \ (\forall)\,(\xi_1,\ldots,\xi_k)\in\mathcal{G}^k,\ \ (\forall)\, 1\le\alpha\le k, \tag{3.6}$$

where $\widehat{J}(\xi_1,\ldots,\xi_k): M\longrightarrow\mathbb{R}, \widehat{J}(\xi_1,\ldots,\xi_k)(x) := J(x)(\xi_1,\ldots,\xi_k)$.

The momentum map $J: M\longrightarrow\mathcal{G}^{*^k}$ is called $(\Phi, Ad^{*^k})$-equivariant if

$$J(\Phi_g(x)) = Ad^{*^k}_{g^{-1}}J(x),\ \ (\forall)\, g\in G,\ \ (\forall)\, x\in M. \tag{3.7}$$

Consider now $J: M\longrightarrow\mathcal{G}^{*^k}$ a $(\Phi, Ad^{*^k})$-equivariant momentum map for the $k$-symplectic action $\Phi: G\times M\longrightarrow M$ of the Lie group $G$ on the $k$-symplectic manifold $(M,\omega_\alpha,\mathcal{F})_{1\le\alpha\le k}$ and $\mu\in\mathcal{G}^{*^k}$ a regular value of $J$. Denote by $G_\mu := \{g\in G \mid Ad^{*^k}_{g^{-1}}(\mu) = \mu\}\subset G$ the isotropy subgroup of $\mu$ with respect to the $k$-coadjoint action. Assume that $G_\mu$ acts freely and properly on $J^{-1}(\mu)$. The reduction theorem for $k$-symplectic manifolds states:

**Theorem 3.2.** [3] *Under the hypotheses above, on the manifold $M_\mu := J^{-1}(\mu)/G_\mu$ there exists a unique $k$-symplectic structure $((\omega_\mu)_\alpha,\mathcal{F}_\mu)_{1\le\alpha\le k}$, such that*

$${\pi_\mu}^*(\omega_\mu)_\alpha = {i_\mu}^*\omega_\alpha,\ \ (\forall)\, 1\le\alpha\le k,$$

*where $\pi_\mu: J^{-1}(\mu)\longrightarrow M_\mu$ is the canonical projection and $i_\mu: J^{-1}(\mu)\longrightarrow M$ the inclusion, if the following condition is satisfied: $(\pi_\mu)_*X\in\ker(\omega_\mu)_\alpha$ implies $(i_\mu)_*X\in\ker\omega_\alpha$, for every $X\in\Gamma(TJ^{-1}(\mu))$ and for all $\alpha\in\{1,\ldots,k\}$.*

The foliation $\mathcal{F}_\mu$ on the reduced manifold $M_\mu$ is constructed, using the reduced 2-forms $(\omega_\mu)_\alpha$, $\alpha\in\{1,\ldots,k\}$, in the following way: take $(L_\mu)_\alpha := \bigcap_{\beta\ne\alpha}\ker(\omega_\mu)_\beta$ and consider $L_\mu := (L_\mu)_1\oplus\cdots\oplus(L_\mu)_k$. So $\mathcal{F}_\mu$ is the foliation integral to the distribution $L_\mu$. Denote by $Q_\mu$ the normal bundle on $\mathcal{F}_\mu$.

Observe that for the reduced manifold $(M_\mu,(\omega_\mu)_\alpha,\mathcal{F}_\mu)_{1\le\alpha\le k}$, the conditions (i) and (ii) in the Theorem 2.1 are also satisfied. Indeed, let $\tilde{Y},\tilde{Y'}\in\Gamma(Q_\mu)$ and $\alpha\in\{1,\ldots,k\}$. Then, since $(\pi_\mu)_*$ is surjective, $(\omega_\mu)_\alpha(\tilde{Y},\tilde{Y'}) = (\omega_\mu)_\alpha((\pi_\mu)_*Y,(\pi_\mu)_*Y') = \omega_\alpha((i_\mu)_*Y,(i_\mu)_*Y')$, with $Y,Y'\in\Gamma(TJ^{-1}(\mu))$.

Since $(\pi_\mu)_*X\in\ker(\omega_\mu)_\alpha$ implies $(i_\mu)_*X\in\ker\omega_\alpha$, for $X\in\Gamma(TJ^{-1}(\mu))$, we obtain that $(i_\mu)_*Y,(i_\mu)_*Y'\in\Gamma(Q)$ [because, if we assume that $(i_\mu)_*Y\in\ker\omega_\alpha$, then, in particular, for any $W\in\Gamma(TJ^{-1}(\mu))$, we have $0=\omega_\alpha((i_\mu)_*Y,(i_\mu)_*Y') = (\omega_\mu)_\alpha((\pi_\mu)_*Y,(\pi_\mu)_*Y')$ and so $(\pi_\mu)_*Y\in\ker(\omega_\mu)_\alpha$ which contradicts the fact that $\tilde{Y} = (\pi_\mu)_*Y\in\Gamma(Q_\mu)$]. It follows that $(\omega_\mu)_\alpha(\tilde{Y},\tilde{Y'}) = 0$, for any $\tilde{Y},\tilde{Y'}\in\Gamma(Q_\mu)$ and $\alpha\in\{1,\ldots,k\}$, which proves (i).

For (ii), let $\tilde{X}\in\Gamma((L_\mu)_\alpha)$, $\tilde{Y}\in\Gamma(Q_\mu)$ and $X,Y\in\Gamma(TJ^{-1}(\mu))$ such that $\tilde{X} = (\pi_\mu)_*X$, $\tilde{Y} = (\pi_\mu)_*Y$. Then $(i_\mu)_*X\in\Gamma(L_\alpha)$, $(i_\mu)_*Y\in\Gamma(Q)$ and hence $[(i_\mu)_*X,(i_\mu)_*Y]\in\Gamma(L_\alpha\oplus Q)$. If $[\tilde{X},\tilde{Y}]\in\Gamma((L_\mu)_\beta)$ [or respectively, $\Gamma(Q_\mu)$ or respectively, $\Gamma((L_\mu)_\beta\oplus Q_\mu)$, $\beta\ne\alpha$], follows that $[(i_\mu)_*X,(i_\mu)_*Y]\in\Gamma(L_\beta)$ [or respectively, $\Gamma(Q)$ or respectively, $\Gamma(L_\beta\oplus Q)$, $\beta\ne\alpha$] which contradicts the fact that $[(i_\mu)_*X,(i_\mu)_*Y]\in\Gamma(L_\alpha\oplus Q)$.

Generalizing the result obtained by I. Vaisman in [7], we shall give a reduction type theorem for the canonical connection on a $k$-symplectic manifold as follows.

Denote by $\mathcal{I}$ the isotropic foliation defined by the connected components of the orbits of $G_\mu$ in $J^{-1}(\mu)$ with the property $T\mathcal{I} = (TJ^{-1}(\mu))^{\perp_{\omega_\alpha}}\cap TJ^{-1}(\mu)$, $(\forall)\ \alpha\in\{1,\ldots,k\}$. Because the canonical connection $\nabla$ is torsion free and $\nabla\omega_\alpha = 0$, $(\forall)\ \alpha\in\{1,\ldots,k\}$, it follows that $T\mathcal{I}$ is parallel with respect to $\nabla$. Assume now that the $k$-symplectic action $\Phi$ is a $\nabla$-*affine action*, that is, it preserves the connection $\nabla$.

**Theorem 3.3.** *Let $(M, \omega_\alpha, \mathcal{F})_{1\leq\alpha\leq k}$ be a $k$-symplectic manifold on which we have a $\nabla$-affine $k$-symplectic action $\Phi$ of a Lie group $G$ and there exists a $(\Phi, Ad^{*^k})$-equivariant momentum map $J : M \longrightarrow \mathcal{G}^{*^k}$. Let $\mu \in \mathcal{G}^{*^k}$ be a regular value of $J$. Assume that the hypotheses of the Theorem 3.2 hold and that $J^{-1}(\mu)$ is $\nabla$-self-parallel. Then the canonical connection $\nabla$ defined in Theorem 2.1 induces a canonical connection $\nabla_\mu$ on $M_\mu = J^{-1}(\mu)/G_\mu$.*

*Proof.* Define

$$\nabla_\mu(\tilde{X}, \tilde{Y}) := (\pi_\mu)_*(\nabla_X Y), \tag{3.8}$$

for $\tilde{X} = (\pi_\mu)_* X$, $\tilde{Y} = (\pi_\mu)_* Y$, with $X, Y \in \Gamma(TJ^{-1}(\mu))$ $G_\mu$-invariant vector fields on $J^{-1}(\mu)$ up to a term in $\Gamma(T\mathcal{I})$.

It is correctly-defined. Indeed, since $J^{-1}(\mu)$ is $\nabla$-self-parallel, the vector field $\nabla_X Y$ is tangent to $J^{-1}(\mu)$, for any $X, Y \in \Gamma(TJ^{-1}(\mu))$. As $X$ and $Y$ are defined up to a term in $\Gamma(T\mathcal{I})$, notice that for $X \mapsto X + Z$ with $Z \in \Gamma(T\mathcal{I})$, we have $(\pi_\mu)_*(\nabla_{X+Z} Y - \nabla_X Y) = (\pi_\mu)_*(\nabla_Z Y) = (\pi_\mu)_*(\nabla_Y Z + [Z, Y]) = (\pi_\mu)_*(\nabla_Y Z) + (\pi_\mu)_*([Z, Y]) = 0$ (because $\nabla$ is torsion free and preserves $T\mathcal{I}$) and respectively, for $Y \mapsto Y + Z$ with $Z \in \Gamma(T\mathcal{I})$, we have $(\pi_\mu)_*(\nabla_X(Y+Z) - \nabla_X Y) = (\pi_\mu)_*(\nabla_X Z) = 0$.

The connection $\nabla_\mu$ satisfies the conditions that define the unique canonical connection on $M_\mu$. Indeed, they follow from the properties of $\nabla$, from the fact that

$$\begin{aligned}[(\nabla_\mu(\omega_\mu)_\alpha)(\tilde{X}, \tilde{Y}, \tilde{Z})](\tilde{x}) &= [(\pi_\mu)_* X((\omega_\mu)_\alpha((\pi_\mu)_* Y, (\pi_\mu)_* Z))](\pi_\mu(x)) \\ &\quad - [(i_\mu)_* X(\omega_\alpha((i_\mu)_* Y, (i_\mu)_* Z))](i_\mu(x)) = 0,\end{aligned}$$

for any $\tilde{X} = (\pi_\mu)_* X$, $\tilde{Y} = (\pi_\mu)_* Y$, $\tilde{Z} = (\pi_\mu)_* Z \in \Gamma(TM_\mu)$, $\tilde{x} = \pi_\mu(x) \in M_\mu$ and $\alpha \in \{1, \ldots, k\}$ and the last one from the fact that $\nabla$ is torsion free. □

## References


[1] Awane, A., *k-symplectic structures*, J. Math. Phys. **33** (1992), 4046-4052

[2] Awane, A., *Some affine properties of the k-symplectic manifolds*, Beitr. Algebra Geom. **39** (1998), 75-83

[3] Blaga, A. M., *The reduction of a k-symplectic manifold*, Mathematica, Cluj-Napoca **50 (73)**, No. 2 (2008), 149-158

[4] Cappelletti Montano, B., Blaga, A. M., *Some geometric structures associated with a k-symplectic manifold*, J. Phys. A: Math. Theor. **41** (2008)

[5] Munteanu, F., Rey, A. and Salgado, M., *The Günther's formalism in classical field theory: momentum map and reduction*, J. Math. Phys. **45** (2004), 1730-1751

[6] Puta, M., *Some remarks on k-symplectic manifolds*, Tensor N. S. **47** (1998), 109-115

[7] Vaisman, I., *Connections under symplectic reduction*, arXiv: math. SG / 0006023 v1 (2000)



West University of Timişoara
Department of Mathematics
V. Pârvan 4, 300223 Timişoara, Romania
*E-mail address*: adara@math.uvt.ro