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Statistical approximation properties of Kantorovich type q-MKZ operators

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ABSTRACT.
In this study we have introduced the Kantorovich type generalization of Meyer-König and Zeller operators based on q-integers. With the help of some recent studies on q-calculus, we have obtained the statistical Korovkin type approximation properties of the operator. We have also examined the order of statistical approximation by means of modulus of continuity.

1. INTRODUCTION

The Meyer-König and Zeller operators (MKZ) were first introduced in 1960 in [15]. A modification of these operators was given by Cheney and Sharma [3] in order to obtain the monotonicity properties. These operators, called as Bernstein power series, are defined as,

$$M_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) m_{n,k}(x), \qquad 0 \le x < 1 \tag{1.1}$$

where

$$m_{n,k}(x) = \left(\begin{array}{c} k+n \\ k \end{array}\right) x^k (1-x)^{n+1}.$$

These operators were generalized by Doğru in [6]. A Stancu type generalization of these operators has been studied by Agratini [1] and Kantorovich type generalization of Agratini's operators were constructed and statistical approximation properties were examined by Doğru, Duman and Orhan in [8]. A Kantorovich type generalization of MKZ operators can also be found in [7].
In 2000, Trif [18] introduced a generalization of MKZ operators based on the q-integers. The q-generalization of the operators (1.1) was defined as,

$$M_{n,q}(f,x) = p_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{[k]}{[k+n]}\right) \left[\begin{array}{c} k+n \\ k \end{array}\right] x^k, \tag{1.2}$$

where

$$p_{n,q}(x) = \prod_{j=0}^{n} (1 - q^j x). \tag{1.3}$$

Trif studied the approximation properties and rate of convergence of the operator $M_{n,q}(f,x)$ for $q \in (0,1]$ and $x \in [0,1)$. Since they will be used in the further sections, we recall the following expressions satisfied by the operator (1.2)

$$M_{n,q}(e_0,x) = 1 \tag{1.4}$$

$$M_{n,q}(e_1,x) = x \tag{1.5}$$

$$0 \le M_{n,q}(e_2,x) - x^2 \le (q-1)x^2 + \frac{x}{[n]} \tag{1.6}$$

where e_ν is defined as monomials $e_\nu(x) : x \to x^\nu$. In order to give explicit formulae for the second moment of the q-MKZ operators, O. Doğru and O. Duman [9] modified the operators as,

$$M_n(f,q,x) = u_{n,q}(x) \sum_{k=0}^{\infty} f\left(\frac{q^n[k]}{[k+n]}\right) \left[\begin{array}{c} k+n \\ k \end{array}\right] x^k, \tag{1.7}$$

and they examined the approximation properties of these operators via A-statistical convergence.

The main aim of this paper is to introduce a Kantorovich type generalization of the q-MKZ operators given in (1.2). Recently, the Kantorovich type generalization of q-Bernstein operators has been defined in [4] and the statistical approximation properties of the operators have been examined in [5] and [17]. Using the similar techniques, here we will examine the statistical approximation properties of Kantorovich type q-MKZ operators and obtain the rate of statistical convergence by means of modulus of continuity.

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2. Construction of operator

We first recall the definition of q-integers. (see, for instance [2], [12]).

Definition 2.1. For any fixed real number $q > 0$, the q-integer $[r]$ is defined as

$$[r]_q := [r] = \begin{cases} \dfrac{1-q^r}{1-q} & q \neq 1, \\ r & q = 1, \end{cases}$$

for all nonnegative integers r.

The q-factorial $[r]!$ and q-binomial $\left[\begin{array}{c} n \\ r \end{array}\right]$, $(n \geq r \geq 0)$ are also defined by

$$[r]! = \begin{cases} [r][r-1]...[1] \ ; & q \neq 1, \\ 1 \ ; & q = 1, \end{cases}$$

and

$$\left[\begin{array}{c} n \\ r \end{array}\right] = \frac{[n]!}{[n-r]![r]!},$$

respectively.

In traditional infinitesimal calculus, which is called as quantum calculus, the definition of derivative does not consist of the notation of limit. Firstly, at this point, let us recall the concepts of q-differential, q-derivative and q-integral respectively.

Definition 2.2. For an arbitrary function $f(x)$, the q-differential is given by

$$d_q f(x) = f(qx) - f(x).$$

Definition 2.3. For an arbitrary function $f(x)$, the q-derivative is defined as

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}.$$

Definition 2.4. [12] Suppose $0 < a < b$. The definite q-integral is defined as

$$\int_0^b f(t) d_q t = (1-q) b \sum_{j=0}^{\infty} f(q^j b) q^j \qquad 0 < q < 1, \tag{2.8}$$

and

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \tag{2.9}$$

Since the definite q-integral in the interval $[a, b]$ is defined by the difference of two infinite sums, its usage causes some problems in obtaining the q-analogues of some well-known integral-inequalities. In order to overcome these problems Marinković et.al. [14] introduced a new type of q-integral. This new q-integral is called as Riemann type q-integral and defined by

$$R_q(f; a, b) := \int_a^b f(x) d_q^R x = (1-q)(b-a) \sum_{j=0}^{\infty} f(a + (b-a)q^j) q^j, \tag{2.10}$$

where a, b and q are some real numbers such that $0 < a < b$ and $0 < q < 1$. Contrary to the classical definition of q-integral (2.9), this definition includes only points within the interval of integration.

It is shown that the Riemann type q-integral is a positive operator and it satisfies the following Hölder's inequality: Let $0 < a < b$, $0 < q < 1$ and $\frac{1}{m} + \frac{1}{n} = 1$. Then

$$R_q(|fg|; a; b) \leq (R_q(|f|^m; a; b))^{\frac{1}{m}} (R_q(|g|^n; a; b))^{\frac{1}{n}}. \tag{2.11}$$

Now we define the Kantorovich type generalization of q-MKZ operators as follows:

$$M_n^*(f; q, x) = p_{n,q}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} k+n \\ k \end{array}\right]_q \left(\frac{x}{q}\right)^k \int_{[k]_q}^{[k+1]_q} f\left(\frac{t}{[k+n]_q}\right) d_q^R t. \tag{2.12}$$

Here

$$p_{n,q}(x) = \prod_{s=0}^{n} (1 - q^s x). \tag{2.13}$$

In the following section we give the statistical approximation properties of the operator (2.12).

3. Statistical convergence properties

Before proceeding further let us recall some notations on the concept of statistical convergence, which was first introduced by H. Fast [10] in 1951 and has recently become an important research area in approximation theory.

Recall that the natural density, δ, of a set $K \subseteq N$ is defined by

$$\delta(K) = \lim_n \frac{1}{n}\{\text{the number } k \leq n : k \in K\}$$

provided the limit exists (see [16]). A sequence $x = (x_k)$ is called statistically convergent to a number L if, for every $\epsilon > 0$

$$\delta\{k : \ |x_k - L| \geq \epsilon\} = 0$$

and it is denoted as $st - \lim_k x_k = L$.

The concept of statistical convergence was used in approximation theory by A. D. Gadjiev and C. Orhan [11]. They proved the Bohman-Korovkin type approximation theorem for statistical convergence as follows:

Theorem A. [11] *If the sequence of linear positive operators $A_n : C[a,b] \to C[a,b]$ satisfies the conditions,*

$$st - \lim_n \|A_n(e_\nu; .) - e_\nu\|_{C[a,b]} = 0 \quad , \quad e_\nu(t) = t^\nu$$

for $\nu = 0, 1, 2$, then for any function $f \in C[a,b]$,

$$st - \lim_n \|A_n(f; .) - f\|_{C[a,b]} = 0.$$

Before giving the Korovkin-type theorem for the operator (2.12), let us give the following Lemmas:

Lemma 3.1. *For all $n \in N$, $x \in [0,a] (0 < a < 1)$ and for $0 < q < 1$, we have;*

$$M_n^*(e_0; q; x) = 1. \tag{3.14}$$

Proof.

$$M_n^*(e_0; q; x) = p_{n,q}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} k+n \\ k \end{array} \right]_q \left(\frac{x}{q}\right)^k \int_{[k]_q}^{[k+1]_q} d_q^R t.$$

Since $\int_{[k]_q}^{[k+1]_q} d_q^R t = q^k$ we immediately get (3.14). □

Lemma 3.2. *For all $n \in N$, $x \in [0,a] (0 < a < 1)$ and for $0 < q < 1$, we have;*

$$0 \leq M_n^*(e_1; q; x) - x \leq \frac{1}{[2]_q} \frac{1}{[n]_q}. \tag{3.15}$$

Proof.

$$M_n^*(e_1; q; x) = p_{n,q}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} k+n \\ k \end{array} \right]_q \left(\frac{x}{q}\right)^k \int_{[k]_q}^{[k+1]_q} \frac{t}{[k+n]_q} d_q^R t.$$

One can easily compute that

$$\int_{[k]_q}^{[k+1]_q} \frac{t}{[k+n]_q} d_q^R t = \frac{[k]_q}{[k+n]_q} q^k + \frac{1}{[k+n]_q} \frac{q^{2k}}{[2]_q}.$$

Substituting this q-integral into the previous equality, we get,

$$M_n^*(e_1; q; x) - x = p_{n,q}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} k+n \\ k \end{array} \right]_q \frac{[k]_q}{[k+n]_q} x^k - x \tag{3.16}$$
$$+ \frac{1}{[2]_q} p_{n,q}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} k+n \\ k \end{array} \right]_q \frac{1}{[k+n]_q} (qx)^k.$$

Since $q^k \leq 1$ for $0 < q < 1$ and $[k+n] \geq [n]$ we can write

$$M_n^*(e_1; q; x) - x \leq M_{n,q}(e_1; x) - x + \frac{1}{[2]_q} \frac{1}{[n]_q} p_{n,q}(x) \sum_{k=0}^{\infty} \left[\begin{array}{c} k+n \\ k \end{array} \right]_q x^k.$$

From (1.5), we finally get

$$M_n^*(e_1; q; x) - x \leq \frac{1}{[2]_q} \frac{1}{[n]_q}.$$

On the other hand it is obvious from (3.16) that $M_n^*(e_1; q; x) - x \geq 0$. □

Lemma 3.3. *For all $n \in N$, $x \in [0,a](0<a<1)$ and for $0<q<1$, we have;*

$$0 \le M_n^*(e_2;q;x) - x^2 \le M_{n,q}(e_2;x) - x^2 + \frac{2}{[2]_q}\frac{1}{[n]_q}M_{n,q}(e_1;x) + \frac{1}{[3]_q}\frac{1}{[n]_q^2}. \tag{3.17}$$

Proof.

$$M_n^*(e_2;q;x) = p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \left(\frac{x}{q}\right)^k \int_{[k]_q}^{[k+1]_q} \frac{t^2}{[k+n]_q^2} d_q^R t$$

Computing the q-integral on the right hand side of the above equality and then making some calculations, we get

$$M_n^*(e_2;q;x) = p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \frac{[k]_q^2}{[k+n]_q^2}x^k$$
$$+\frac{2}{[2]_q}p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \frac{[k]_q}{[k+n]_q^2}q^k x^k + \frac{1}{[3]_q}p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \frac{1}{[k+n]_q^2}q^{2k}x^k$$

that is,

$$M_n^*(e_2;q;x) - x^2 = M_{n,q}(e_2;x) - x^2 + \frac{2}{[2]_q}p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \frac{[k]_q}{[k+n]_q^2}q^k x^k$$
$$+\frac{1}{[3]}p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \frac{1}{[k+n]_q^2}q^{2k}x^k. \tag{3.18}$$

Again using the inequality

$$\frac{q^k}{[k+n]} \le \frac{1}{[n]} \qquad 0 \le q < 1, \quad k = 0,1,2,...$$

we can write

$$M_n^*(e_2;q;x) - x^2 \le M_{n,q}(e_2;x) - x^2 + \frac{2}{[2]_q}p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \frac{[k]_q}{[k+n]_q}\frac{1}{[n]_q}x^k$$
$$+\frac{1}{[3]}p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \frac{1}{[n]_q^2}x^k.$$

Therefore we have,

$$M_n^*(e_2;q;x) - x^2 \le M_{n,q}(e_2;x) - x^2 + \frac{2}{[2]_q}\frac{1}{[n]_q}M_{n,q}(e_1;x) + \frac{1}{[3]_q}\frac{1}{[n]_q^2}. \tag{3.19}$$

On the other hand, since $M_{n,q}(e_2;x) - x^2 > 0$, from (3.18) we can write

$$M_n^*(t^2;q;x) - x^2 \ge 0$$

from which the proof of Lemma 3 is completed. □

Now using the above Lemmas we can give the following statistical approximation theorem for the operator $M_n^*(f;q;x)$.

Theorem 3.1. *Let $q := (q_n)$, $0 < q_n < 1$, be a sequence satisfying*

$$st - \lim_n q_n = 1 \quad \text{and} \quad st - \lim_n \frac{1}{[n]_{q_n}} = 0. \tag{3.20}$$

Then for all $f \in C[0,a]$ with $0 < a < 1$, the operator $M_n^*(f;q;x)$ satisfies

$$st - \lim_n \|M_n^*(f;q_n,.) - f(.)\|_{C[0,a]} = 0. \tag{3.21}$$

Proof. From the definition of Riemann type q-integral, it can be easily seen that $M_n^*(f;q;x)$ is a linear-positive operator. So if we can show that, for $i = 0,1,2$

$$st - \lim_n \|M_n^*(e_i;q_n;.) - x^i\|_{C[0,a]} = 0, \tag{3.22}$$

then the proof follows from Theorem A.

For $i = 0$, it is clear from Lemma 1 that

$$st - \lim_n \|M_n^*(e_0;q_n;.) - 1\|_{C[0,a]} = 0. \tag{3.23}$$

For $i = 1$, we have

$$\|M_n^*(e_1;q_n,x) - x\|_{C[0,a]} \le \frac{1}{[2]_{q_n}}\frac{1}{[n]_{q_n}} \tag{3.24}$$

from Lemma 2. Now for a given $\epsilon > 0$, let us define the following sets:

$$T := \{k : \|M_k^*(e_1; q_k; .) - x\|_{C[0,a]} \geq \epsilon\},$$

$$T_1 := \{k : \frac{1}{[2]_{q_k}} \frac{1}{[k]_{q_k}} \geq \epsilon\}.$$

From (3.24) it is clear that $T \subseteq T_1$. So we can write,

$$\delta\{k \leq n : \ \|M_k^*(e_1; q_k; .) - x\|_{C[0,a]} \geq \epsilon\} \leq \delta\{k \leq n : \frac{1}{[2]_{q_k}} \frac{1}{[k]_{q_k}} \geq \epsilon\}.$$

From the conditions (3.20), the right hand side of the above inequality is zero. Therefore we have,

$$\delta\{k \leq n : \ \|M_k^*(e_1; q_k; .) - x\|_{C[0,a]} \geq \epsilon\} = 0$$

which implies

$$st - \lim_n \|M_n^*(e_1; q_n; .) - x\|_{C[0,a]} = 0. \tag{3.25}$$

Lastly for $i = 2$ we can write

$$\begin{aligned} \|M_n^*(e_2; q_n; .) - x^2\|_{C[0,a]} \leq & \|M_{n,q_n}(e_2; .) - x^2\|_{C[0,a]} \\ & + \frac{2}{[2]_{q_n}} \frac{1}{[n]_{q_n}} \|M_{n,q_n}(e_1; .)\|_{C[0,a]} + \frac{1}{[3]_{q_n}} \frac{1}{[n]_{q_n}^2} \\ \leq & \|M_{n,q_n}(e_2; .) - x^2\|_{C[0,a]} \\ & + 2\left(\frac{1}{[n]_{q_n}} \|M_{n,q_n}(e_1; .)\|_{C[0,a]} + \frac{1}{[n]_{q_n}^2}\right). \end{aligned} \tag{3.26}$$

Now, for a given $\epsilon > 0$, let us define the following sets:

$$\begin{aligned} K &:= \left\{k : \|M_k^*(e_2; q_k; .) - x^2\|_{C[0,a]} \geq \epsilon\right\}, \\ K_1 &:= \left\{k : \|M_{k,q_k}(e_2; .) - x^2\|_{C[0,a]} \geq \frac{\epsilon}{3}\right\} \\ K_2 &:= \left\{k : \frac{1}{[k]_q} \|M_{k,q_k}(e_1; .)\|_{C[0,a]} \geq \frac{\epsilon}{6}\right\} \\ K_3 &:= \left\{k : \frac{1}{[k]_q^2} \geq \frac{\epsilon}{6}\right\} \end{aligned}$$

From (3.26) it is clear that $K \subseteq K_1 \cup K_2 \cup K_3$. Therefore we have,

$$\delta\{K\} \leq \delta\{K_1\} + \delta\{K_2\} + \delta\{K_3\}. \tag{3.27}$$

Taking the conditions given in (3.20) into account, one can easily show from (1.5) and (1.6) that $st - \lim_n \frac{1}{[n]_q} \|M_{n,q_n}(e_1; .)\|_{C[0,a]} = 0$ and $st - \lim_n \|M_{n,q_n}(e_2; x) - x^2\|_{C[0,a]} = 0$. Therefore the right hand side of (3.27) becomes zero and hence we get

$$\delta\{k : \|M_k^*(e_2; q_k; .) - x^2\|_{C[0,a]} \geq \epsilon\} = 0$$

i.e.,

$$st - \lim_n \|M_n^*(e_2; q_n; .) - x^2\|_{C[0,a]} = 0. \tag{3.28}$$

Now by (3.23), (3.25) and (3.28) we conclude from Theorem A that for all $f \in C[0, a]$

$$st - \lim_n \|M_n^*(f; q_n; .) - f\|_{C[0,a]} = 0.$$

□

Remark 3.1. For example, if we choose (q_n) as

$$q_n = \begin{cases} \frac{1}{2} & n = m^2 \\ 1 - \frac{1}{n} & n \neq m^2, \end{cases}$$

then $0 < q_n < 1$ and the conditions given in (3.20) are satisfied.

4. Rates of statistical convergence

In this section, we compute the approximation order of the operator $M_n^*(f;q;x)$ by means of modulus of continuity.

The modulus of continuity of f, $w(f,\delta)$, is defined by

$$w(f;\delta) = \sup_{\substack{|x-y|\leq\delta \\ x,y\in[0,a]}} |f(x)-f(y)|. \tag{4.29}$$

It is well-known that, for a function $f \in C[0,a]$,

$$\lim_{\delta\to 0^+} w(f;\delta) = 0 \tag{4.30}$$

and for any $\delta > 0$

$$|f(x)-f(y)| \leq w(f;\delta)\left(\frac{|x-y|}{\delta}+1\right). \tag{4.31}$$

Before giving the theorem on the rate of convergence of the operator $M_n^*(f;q;x)$, let us first examine its second moment:

$$M_n^*((e_1-x)^2;q;x) = M_n^*(e_2;q;x) - x^2 - 2x(M_n^*(e_1;q;x)-x) \tag{4.32}$$

$$\begin{aligned} &\|M_n^*((e_1-x)^2;q;x)\|_{C[0,a]} \\ &\leq \|M_n^*(e_2;q;x)-x^2\|_{C[0,a]} + 2\|x\|\|M_n^*(e_1;q;x)-x\|_{C[0,a]} \end{aligned} \tag{4.33}$$

Using Lemma 3.2 and Lemma 3.3 we can write,

$$\begin{aligned} \|M_n^*((e_1-x)^2;q;x)\|_{C[0,a]} &\leq \|M_{n,q}(e_2;x)-x^2\|_{C[0,a]} \\ &+\frac{2}{[2]_q}\frac{1}{[n]_q}\|M_{n,q}(e_1;x)\|_{C[0,a]} + \frac{1}{[3]_q}\frac{1}{[n]_q^2} + \frac{2a}{[2]_q}\frac{1}{[n]_q} \end{aligned}$$

Since

$$\|M_{n,q}(e_1;x)\|_{C[0,a]} = a$$

from (1.5) and

$$\|M_{n,q}(e_2;x)-x^2\|_{C[0,a]} \leq (1-q)a^2 + \frac{a}{[n]_q}$$

from (1.6), we have,

$$\|M_n^*((e_1-x)^2;q;x)\|_{C[0,a]} \leq (1-q)a^2 + \left(a+\frac{4a}{[2]_q}\right)\frac{1}{[n]_q} + \frac{1}{[3]_q}\frac{1}{[n]_q^2}. \tag{4.34}$$

The following theorem gives the rate of convergence of the operator $M_n^*(f;q;x)$ to the function $f(x)$ by means of modulus of continuity.

Theorem 4.2. *If the sequence $q := (q_n)$ satisfies the condition given in (3.20), then*

$$\|M_n^*(f;q_n;.)-f\|_{C[0,a]} \leq 2w(f,\delta_n) \tag{4.35}$$

for all $f \in C[0,a]$, where

$$\delta_n = \sqrt{(1-q_n)a^2 + \left(a+\frac{4a}{[2]_{q_n}}\right)\frac{1}{[n]_{q_n}} + \frac{1}{[3]_{q_n}}\frac{1}{[n]_{q_n}^2}}. \tag{4.36}$$

Proof. We shall use Popoviciu's technique (see, for instance [13]). Let $f \in C[0,a]$. Using the linearity and positivity of $M_n^*(f;q;x)$ and then applying (4.31), we get

$$\begin{aligned} |M_n^*(f;q;x)-f(x)| &\leq M_n^*(|f(t)-f(x)|;q;x) \\ &\leq w(f,\delta)\{1+\frac{1}{\delta}M_n^*(|t-x|;q;x)\} \end{aligned}$$

for all $n \in N$ and $x \in [0, a]$. Writing the operator M_n^* explicitly and then applying Hölder's inequality to the q-integral, we get

$$|M_n^*(f;q,x) - f(x)| \leq w(f,\delta)\left\{1 + \frac{1}{\delta}p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \left(\frac{x}{q}\right)^k \right.$$
$$\left. \left(\int_{[k]_q}^{[k+1]_q}\frac{(t-x)^2}{[k+n]_q^2}d_q^R t\right)^{1/2}\left(\int_{[k]_q}^{[k+1]_q}d_q^R t\right)^{1/2}\right\}$$
$$= w(f,\delta)\left\{1 + \frac{1}{\delta}\left(p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \left(\frac{x}{q}\right)^k \int_{[k]_q}^{[k+1]_q}\frac{(t-x)^2}{[k+n]_q^2}d_q^R t\right)^{1/2}\right.$$
$$\left.\left(p_{n,q}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} k+n \\ k \end{array}\right]_q \left(\frac{x}{q}\right)^k \int_{[k]_q}^{[k+1]_q}d_q^R t\right)^{1/2}\right\}.$$

Since the expression in the 2^{nd} parenthesis of the above inequality equals 1, we have

$$|M_n^*(f;q,x) - f(x)| \leq w(f,\delta)\left\{1 + \frac{1}{\delta}(M_n^*((t-x)^2;q;x))^{1/2}\right\}.$$

Now taking q as a sequence (q_n) satisfying the conditions given in (3.20) and then taking maximum of both sides on $[0, a]$, we get

$$\|M_n^*(f;q_n,.) - f\|_{C[0,a]}$$
$$\leq w(f,\delta)\left\{1 + \frac{1}{\delta}[(1-q_n)a^2 + \left(a + \frac{4a}{[2]_{q_n}}\right)\frac{1}{[n]_{q_n}} + \frac{1}{[3]_{q_n}}\frac{1}{[n]_{q_n}^2}]^{1/2}\right\}$$

from (4.34). Now choosing $\delta := \delta_n$ as in (4.36), we get

$$\|M_n^*(f;q_n,.) - f\|_{C[0,a]} \leq 2w(f,\delta_n)$$

and the proof is completed. □

Remark 4.2. From the conditions (3.20) one can see that

$$st - \lim_n M_n^*((e_1 - x)^2; q_n; x) = 0.$$

This implies $st - \lim_n w(f,\delta_n) = 0$ from (4.30). Hence Theorem 4.2 gives us the rate of statistical convergence of the operator $M_n^*(f;q;x)$ to the function $f(x)$.

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