Weak near-rings

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Abstract.

We define a generalization of the concept of rings, weak near-rings. This is a particular case of infra-near-rings defined by M. Stefănescu and generalizes also the concept of weak rings (A. Climescu, 1964). Correspondences between a class W of weak near-rings and a class of near-rings, and between W and a class of groups are established in the paper. In this context, we extend some results obtained for nonassociative rings by Mal'cev and Weston, and for near-rings, by M. Stefănescu.

1. INTRODUCTION

In 1960, A. I. Mal'cev [3] constructed a correspondence between the class of nonassociative rings with identity and a class of axiomatizable nilpotent groups. In his paper, he showed that this correspondence induces an equivalence between their formalized theories (in the Kleene's sense). Eight years later, K. Weston [7] obtained a similar result for the class of nonassociative rings, with or without identity.

In 1977, M. Ştefănescu [5] constructed such a correspondence for distributive near-rings with additively central products and a special class of nilpotent groups. This correspondence, restricted to the subclass of rings is this of Weston. Some remarks on functorial aspects of this correspondence are made in this paper.

We consider here a new notion, that of *weak near-rings*, and we construct such a correspondence for the class of weak near-rings with a class of groups.

We recall first some notions in the theory of near-rings and infra-near-rings. For other properties, see Pilz [4] and Ştefănescu [6].

Definition 1.1. A **right near-ring** is a triple $(N, +, \cdot)$, where *N* is a nonempty set and + and \cdot are binary compositions on *N*, such that :

- (i) (N, +) is a group;
- (ii) (N, \cdot) is a semigroup;

(iii) For all $a, b, c \in N$, $(a+b) \cdot c = a \cdot c + b \cdot c$.

If, for all $a \in N$, we have,

(iv) $a \cdot 0 = 0$,

then *N* is a **0-symmetric right near-ring**.

An element $d \in N$ for which

(v) $d \cdot (a+b) = d \cdot a + d \cdot b$, for all $a, b \in N$,

is called a **distributive element** of *N*.

If *N* satisfies (i), (ii), (iii) and (v), then *N* is a **distributive near-ring**.

As examples of right near-rings we may recall:

- (1) M(G), the near-ring of functions on a (noncommutative) group (G, +) with respect to pointwise addition and mapping composition.
- (2) $M_0(G) = \{f \in M(G) | f(0) = 0\}$ is a 0-symmetric right near-ring, having the endomorphisms of (G, +) as the distributive elements.
- (3) $Aff(A) = \{f = e + c | e \in End(A, +), c \in A\}$; the "affine" mappings on an Abelian group (A, +), is a right near-ring which is not 0-symmetric.

Definition 1.2. A weak ring is a triple $(R, +, \circ)$ such that:

- (i) (R, +) is an Abelian group;
- (ii) (R, \circ) is a semigroup;
- (iii) \circ is right and left weak distributive with respect to the addition, i.e.

$$a \circ (b+c) + a = a \circ b + a \circ c,$$

$$(a+b) \circ c + c = a \circ c + b \circ c,$$

for all $a, b, c \in R$.

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Al. Climescu has studied weak rings and their generalization, by considering weak distributivity of the type:

$$a \circ (b+c) = a \circ b - \varphi(a) + a \circ c,$$

$$(a+b) \circ c = a \circ c - \psi(c) + b \circ c,$$

for all $a, b, c \in R$ and φ, ψ are mappings on R with some properties.

He showed that, for a ring $(R, +, \cdot)$, there is a weak ring structure $(R, +, \circ)$ on R, such that

 $x \circ y = x \cdot y + x + y$, for all $x, y \in R$.

We shall use this idea in that follows.

Definition 1.3. A left infra-near-ring is an ordered triple $(N, +, \cdot)$ such that:

- (i) (N, +) is a group;
- (ii) (N, \cdot) is a semigroup;
- (iii) For all $x, y, z \in N$, $\overline{x} \cdot (y+z) = x \cdot y x \cdot 0 + x \cdot z$.

Examples of infra-near-rings and their properties can be found in M. Stefănescu [6].

First we introduce the notion of weak near-ring and we study the correspondence between the class of weak near-rings and the class of distributive near-rings with the products in the additive center.

In the third section of the paper, we construct the correspondence between the class of weak rings and a class of groups (with some properties).

2. WEAK NEAR-RINGS AND THEIR RELATIONSHIP WITH NEAR-RINGS

We consider the class \mathcal{D} of distributive near-rings N with the property:

(I)
$$ab + c = c + ab, \forall a, b, c \in N.$$

These near-rings are in close proximity of the rings. It is known that, in distributive near-rings, each two products commute:

(II)
$$xy + zw = zw + xy, \forall x, y, z, w \in N.$$

Therefore (I) asks only the commutativity for each product with each element in N.

Remark 2.1. If in a distributive near-ring *N* the identity (I) holds, then the additive center of N, Z = Z(N, +), is an ideal of *N*.

Indeed, *Z* is an additive normal subgroup and for all $c \in Z$ and $x \in N$, $cx, xc \in Z$.

Definition 2.4. The ordered triple $(N, +, \circ)$ is called a **weak near-ring**, if:

- (i) (N, +) is a group;
- (ii) (N, \cdot) is a semigroup;
- (iii) For all $a, b, c \in N$,
 - (1) $a \circ (b+c) = a \circ b a + a \circ b$,
 - (2) $(a+b) \circ c = a \circ c c + b \circ c$.

We may talk also about right (left) weak rings. The following relations can be verified quite easily :

- (iv) $x \circ 0 = x$, for all $x \in N$;
- (v) $0 \circ x = x$, for all $x \in N$;
- (vi) $x \circ (-y) = x x \circ y + x$, for all $x, y \in N$;
- (vii) $(-x) \circ y = y x \circ y + y$, for all $x, y \in N$;

Proposition 2.1. If $(N, +, \cdot)$ is a distributive near-ring satisfying (I) (hence in the class \mathcal{D}), then the multiplication

$$x \circ y := x \cdot y + x + y$$
, for all $x, y \in N$

endows N with a structure $(N, +, \circ)$ of weak near-ring. Conversely, if $(N, +, \circ)$ is a weak near-ring satisfying the condition

$$x \circ y - y - x \in Z(N, +), \forall x, y \in N,$$

then, defining the multiplication

 $x \cdot y := x \circ y - y - x, \forall x, y \in N,$

we obtain a distributive near-ring $(N, +, \cdot)$ in the class \mathcal{D} .

(We denote the class of such weak near-rings by W.)

Proof. We verify the associativity for the composition "°". We have:

 $(x \circ y) \circ z = xyz + xz + yz + xy + x + y + z$

$$x \circ (y \circ z) = xyz + xy + xz + x + yz + y + z;$$

the two right sides are equal, because of (I). Now we verify one of the weak distributivities:

$$x \circ (y + z) = x(y + z) + x + y + z = xy + xz + x + y + z$$

 $x \circ y - x + x \circ z = xy + x + y - x + xz + x + z =$

$$= xy + xz + x + y - x + x + z = xy + xz + x + y + z.$$

We may verify that, for all $x, y \in N$, $x \circ y - y - x$ commutes with all elements. Indeed,

$$x \circ y - y - x + c = xy + x + y - y - x + c = xy + c$$
 (= c + xy)

and

$$c + x \circ y - y - x = c + xy + x + y - y - x = c + xy.$$

Now, if $(N, +, \circ)$ is a weak near-ring with the elements of the form $x \circ y - y - x$ additively central, then $(N, +, \cdot)$ is a distributive near-ring with central products, as we may verify straightforward.

The ideals in a weak near-ring $(N, +, \circ)$ are defined as a normal subgroup A of (N, +) with the properties

 $x \circ a - x, \quad a \circ x - x \in A,$

for all $a \in A$ and $x \in N$.

Then there is a correspondence between ideals in $(N, +, \cdot)$ and $(N, +, \circ)$.

We give now **an example** of a distributive near-ring in the class \mathcal{D} :

Let *R* be a ring, (G, +) be a group (noncommutative). Then $N = R \times G$, with componentwise addition and the multiplication given by

$$(x,g) \circ (y,h) := (xy,0)$$

is a near-ring in the class \mathcal{D} .

Another example of a distributive near-ring satisfying (I) which is not a ring is the following: Let (N, +) be a metabelian group (i.e. with all commutators in the center). Then, defining the multiplication by

$$a \cdot b := [a, b] = -a - b + a + b$$

for $a, b \in N$, we get $(N, +, \cdot)$ a near-ring which is not a ring, is distributive and with additively central products. Let us note that the weak near-rings are left and right infra-near-rings N for which

$$x \cdot 0 = 0 \cdot x = x,$$

for all $x \in N$.

3. Correspondence between $\mathcal W$ and a class of groups

The weak near-rings in the class W as objects and the homomorphisms among weak near-rings as morphism form a category whose full subcategory is that of weak rings.

We denote this category also by W.

Definition 3.5. We denote by \mathcal{G} the class of groups (G, +) which satisfy the following conditions:

- (i) There exists two endomorphisms of *G*, α and β , such that $\alpha \circ \alpha = \beta \circ \beta = \alpha \circ \beta = \beta \circ \alpha = 0$ (the null endomorphism of *G*).
- (ii) Denoting by *K* the intersection of ker α and ker β , then there exists two group homomorphisms, $\tilde{\alpha} : K \to \ker \beta$, $\tilde{\beta} : K \to \ker \alpha$, such that $\tilde{\alpha} \circ \alpha = \beta \circ \tilde{\beta} = \mathbf{1}_K$.
- (iii) The elements of K and those of $Im\tilde{\alpha}$ commutes. The same holds for the elements of K and the elements in $Im\tilde{\beta}$.

It is obvious from Definition 3.1 that K and the group in \mathcal{G} satisfy the next corollaries.

Corollary 3.1. With the above notations $(\alpha \circ \hat{\beta})(x) = (\beta \circ \tilde{\alpha})(x) = 0$.

Corollary 3.2. For $x, y \in K$, the element

$$[-\hat{\beta}(x), \tilde{\alpha}(y)] = \hat{\beta}(x) - \tilde{\alpha}(y) - \hat{\beta}(x) + \tilde{\alpha}(y)$$

belongs to K.

Corollary 3.3. The class of objects $(G, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ with $G \in \mathcal{G}$ and $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ defined in 3.1, together with group homomorphisms $\varphi : G \to G'$, $(G', \alpha', \beta', \tilde{\alpha}', \tilde{\beta}')$ being another object, such that

$$\begin{aligned} \alpha' \circ \varphi &= \varphi \circ \alpha, \qquad \beta' \circ \varphi &= \varphi \circ \beta, \\ \tilde{\alpha}' \circ \varphi/K &= \varphi \circ \tilde{\alpha}, \qquad \tilde{\beta}' \circ \varphi/K &= \varphi \circ \tilde{\beta}, \end{aligned}$$

where φ/K is the restriction of φ to K, forms a category \mathcal{G} .

We can establish our correspondence.

Proposition 3.2. (i) If N is a weak near-ring in W, then $G = N \times N \times N$ can be endowed with a group structure by taking the composition:

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (y_1 + x_1, y_2 + x_2, y_3 + x_2 \circ y_1 - y_1 - x_2 + x_3)$$

(ii) If $(G, \alpha, \beta, \tilde{\alpha}, \tilde{\beta})$ belongs to $\tilde{\mathcal{G}}$, then K, endowed with the operations:

$$\begin{cases} x \oplus y = y + x, \\ x \odot y = [-\tilde{\beta}(y), \tilde{\alpha}(x)] + y + x, \end{cases}$$

is a weak near-ring in W.

Proof. (*i*) Taking $x, y, z \in G$, for the sums (x + y) + z and x + (y + z), the first two components are equal, since (N, +) is a group. For the third components, we have :

$$E_1 = z_3 + (y_2 + x_2) \circ z_1 - z_1 - x_2 - y_2 + y_3 + x_2 \circ y_1 - y_1 - x_2 + x_3,$$

$$E_2 = z_3 + y_2 \circ z_1 - z_1 - y_2 + y_3 + x_2 \circ (z_1 + y_1) - y_1 - z_1 - x_2 + x_3.$$

Using the fact that $(y_2 + x_2) \circ z_1 - z_1 - (y_2 + z_2)$, $x_2 \circ z_1 - z_1 - x_2$, $-x_2 + x_2 \circ y_1 - y_1$ are in the center of (N, +), we get $E_1 = E_2$, and the defined addition is associative.

The neutral element is (0, 0, 0) and the symmetric of *x* is

$$-x = (-x_1, -x_2, -x_3 + x_2 - x_1 - x_2 + x_2 \circ x_1 - x_2).$$

Now we define $\alpha, \beta \in End(G)$ by $\alpha(x) = (0, 0, x_2), \beta(x) = (0, 0, x_1)$. Then

$$\ker \alpha = \{(x_1, 0, x_3) | x_1, x_3 \in N\} \text{ and } \ker \beta = \{(0, x_2, x_3) | x_2, x_3 \in N\},\$$

 $K = \{(0, 0, x_3) | x_3 \in N\}$. The group homomorphisms $\tilde{\alpha} : K \to \ker \beta$ and $\tilde{\beta} : K \to \ker \alpha$ are defined by

$$\tilde{\alpha}((0,0,x_3)) = (0,x_3,0), \qquad \hat{\beta}((0,0,x_3)) = (x_3,0,0)$$

The conditions for α , β , $\tilde{\alpha}$, $\tilde{\beta}$ are verified by straightforward calculations.

Therefore we get the correspondence for a weak near-ring in W to a group in $\tilde{\mathcal{G}}$.

(*ii*) Take G in $\hat{\mathcal{G}}$. We verify that the two operations structures K as a weak near-ring in \mathcal{D} . Indeed, (K, \oplus) is a group, obviously. Now, the second operation is well-defined, by Corollary 3.3. It is associative, as we may verify directly. The left weak distributivity is done because of:

$$\begin{aligned} x \odot (y \oplus z) &= \left[-\tilde{\beta}(z+y), \tilde{\alpha}(x) \right] + z + y + x \\ (x \odot z) \ominus x \oplus x \odot z &= (x \odot z) - x + (x \odot y) = \\ &= \left[-\tilde{\beta}(z), \tilde{\alpha}(x) \right] + z + x - x + \left[-\tilde{\beta}(y), \tilde{\alpha}(x) \right] + y + x = \\ &= \left[-\tilde{\beta}(z), \tilde{\alpha}(x) \right] + \left[-\tilde{\beta}(y), \tilde{\alpha}(x) \right] + z + y + x. \end{aligned}$$

In the same way, we may verify the right weak distributivity law.

Remark 3.2. Since $x, y \in K$ commute with each element in $\tilde{\alpha}(K)$ and $\tilde{\beta}(K)$, we have also:

$$x \odot y = y + \left[-\tilde{\beta}(y), \tilde{\alpha}(x)\right] + x = y + x + \left[-\tilde{\beta}(y), \tilde{\alpha}(x)\right].$$

Corollary 3.4. If (N, +) is metabelian, then G in Proposition 3.5 (ii) is an additive nilpotent group of index at most 3. Conversely, if G is a nilpotent group of index 3, then (K, \oplus) is metabelian.

Proof. For $x, y \in G$, we have:

$$[x, y] = -x - y + x + y = (0, 0, [-y_3, -x_3]),$$

and $[x, y] \in Z(G)$.

Conversely, if *G* is nilpotent of index 3, then (K, \oplus) is metabelian, by straightforward verification.

We consider now the mappings between classes W and $\tilde{\mathcal{G}}$:

$$F: \mathcal{W} \to \mathcal{G}, \qquad H: \mathcal{G} \to \mathcal{W}$$

where F(N) = G (in Proposition 3.5) and H(G) = K (in definition of the class \mathcal{G}). Then HF(N) = K and $FH(G) = F(K) = K^3$.

The mappings

$$\tau: N \to HF(N), \tau(x) = (0, 0, x), \quad \text{for } x \in G,$$

and

$$\sigma: FH(G) \to G, \sigma(x_1, x_2, x_3) = \tilde{\beta}(x_1) + \tilde{\alpha}(x_2) + x_3$$

for $(x_1, x_2, x_3) \in K^3$ (hence $x_i \in K, i = 1, 2, 3$), and all $\tilde{\beta}(x_1), \tilde{\alpha}(x_2), x_3 \in G$ (since $K \subseteq G, \alpha, \beta \in End(G)$).

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We prove that τ is a weak near-ring isomorphism and σ is a group isomorphism. Indeed, $\tau(x+y) = (0, 0, y+x) = \tau(x) \oplus \tau(y)$.

$$\begin{aligned} \tau(x) \odot \tau(y) &= [-\dot{\beta}((0,0,y)), \tilde{\alpha}((0,0,x))] + (0,0,x) + (0,0,y) \\ &= (0,0,xy - y - x + x + y) = (0,0,xy) = \tau(x \cdot y) \end{aligned}$$

 τ is obviously surjective and injective. For τ , $\tau(x + y) = \tau(x) + \tau(y)$, since

$$\beta(y_1 \oplus x_1) + \tilde{\alpha}(y_2 \oplus x_2) + (y_3 \oplus x_2 \odot y_1 \oplus x_2 \oplus x_3) \\ = \tilde{\beta}(x_1) + \tilde{\beta}(y_1) + \tilde{\alpha}(x_2) + \tilde{\alpha}(y_2) + x_3 - x_2 - y_1 + [-\tilde{\beta}(y_1), \tilde{\alpha}(x_2)] + y_1 + x_2 + y_3$$

and

$$\sigma(x) + \sigma(y) = \tilde{\beta}(x_1) + \tilde{\alpha}(x_2) + x_3 + \tilde{\beta}(y_1) + \tilde{\alpha}(y_2) + y_3$$

are equal because of the properties of the groups in the class $\tilde{\mathcal{G}}$.

 σ is a surjection, since, for all $x \in G$, taking $x_1 = \alpha(x), x_2 = \beta(x)$ and

$$x_3 = -\tilde{\alpha}(x_1) - \tilde{\beta}(x_2) + x_1$$

we get $\sigma((x_1, x_2, x_3)) = x$.

If $\sigma(x) = 0$, we have $\tilde{\beta}(x_1) + \tilde{\alpha}(x_2) + x_3 = 0$, and, by applying α and β , we obtain $x_1 = x_2 = 0$, hence $x_3 = 0$ and x = (0, 0, 0).

This means that the next proposition is true.

Proposition 3.3. The categories \tilde{W} and \tilde{G} are equivalent. Moreover, the theories of the classes W and \tilde{G} are syntactic equivalent.

Proof. We prove only the second statement; the first one have been justified before.

The standard formalized theories \mathcal{T}_{W} and $\mathcal{T}_{\mathcal{G}}$ of \mathcal{W} and \mathcal{G} in the sense of [2] have as primitive symbols:

- for $W : \{+, \cdot, 0\};$
- for $\mathcal{G}: \left\{+, 0, \alpha(), \beta(), \tilde{\alpha}(), \tilde{\beta}(), [,]\right\}$,

for denoting algebraic operations, neutral elements, additive operators as unary predicates, commutator brackets. Denote by ''-x'' the element which verifies the equalities

$$x + (-x) = 0 = (-x) + x_{1}$$

in *N* and in *G*. We have also a formula in the theory of \mathcal{G} , " $x \in \ker \alpha \cap \ker \beta$ ", denoted by *P*. A recursive mapping

 $\bar{T}:\mathcal{T}_{\mathcal{W}}\to\mathcal{T}_{\mathcal{G}}$

is defined in the following manner:

For a formula A in \mathcal{T}_{W} , we take $\overline{T}(A) = \tilde{A}$, obtained by replacing x + y by y + x, 0 by 0, $x \cdot y$ by $[-\tilde{\beta}(y), \tilde{\alpha}(x)]$, which is a formula of \mathcal{T}_{G} .

By taking the relativized formula of A with respect to P (see [2], 1.5, page 25), we obtain a formula $A^{(P)}$. Then by the first part of this proposition, A is true in $N \in W$ if and only if $\tilde{A}^{(P)}$ is true in $F(N) \in \mathcal{G}$.

For the converse, taking a formula *B* of $T_{\mathcal{G}}$, this is transformed into its prenex form, $B = (Q_1 x_1)...(Q_n x_n)B_1(x_1,...,x_n,0)$, where Q_i are quantifiers and B_1 is a formula without quantifiers.

We construct H(G) and we take T'(B) in \mathcal{T}_{W} , by using the following replacements:

• $(Q_i x_i)$ is replaced by $(Q_i x_i)(Q_i y_i)(Q_i z_i), i = 1, ..., n;$

•
$$x_i + x_j = x_k$$
 is replaced by $(x_i + x_i = x_k) \land (y_j + y_i = y_k) \land (z_j + y_i x_j + z_i = z_k)$

Then *B* is true on *G* if and only if T'(B) is true on H(G).

Therefore $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{T}_{\mathcal{G}}$ are syntactically equivalent.

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