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# A note on ordered vector spaces and Kantorovich Extension Theorem

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ABSTRACT.

Nontrivial examples that demonstrate the independence of the ordered vector spaces are given and the Kantorovich Extension Theorem is generalized.

## 1. EXAMPLE

A partial order on a set is a reflexive, transitive, antisymmetric relation. An ordered vector space is an ordered pair $(V, \leq)$ where $V$ is a vector space over $\mathbf{R}$, $\leq$ is a partial order on $V$ which is compatible with the structure of $V$ as vector space over $\mathbf{R}$. To be precise, $(V, \leq)$ satisfies the following axioms:

(a) For any $x, y, z \in V$, $x \leq y$ implies $x + z \leq y + z$
(b) For any $\alpha \in \mathbf{R}$ with $\alpha \geq 0$, $0 \leq x$ implies $0 \leq \alpha x$
The purpose of this short note is to provide nontrivial examples that demonstrate the independence of the above axioms. To the best my knowledge, such examples are presently not available in the literature (see the references).

Let $V$ be a nonzero vector space. Choose and fix $0 \neq t \in E$ and define

$$x \leq_a y \Longleftrightarrow \quad y - x = nt \quad \text{for some non-negative integer n}$$

and

$$x \leq_b y \Longleftrightarrow x = y \quad \text{or} \quad (x = 0 \quad \text{and} \quad y = \alpha t \quad \text{for some} \quad 0 \leq \alpha \in \mathbf{R})$$

It is routine to check that the orders $\leq_a$ and $\leq_b$ are partial orders on $V$. Also we have for each $x, y$ and $z \in E$

$$x \leq_a y \Leftrightarrow y - x = nt \Leftrightarrow (y + z) - (x + z) = nt \Leftrightarrow x + z \leq_a y + z$$

and

$$0 \leq_a t \quad \text{and} \quad 0 \not\leq_a (1/2)t$$

For the order $\leq_b$, let $0 \leq \beta \in \mathbf{R}$ and $x \neq y \in E$. Then

$$x \leq_b y \Rightarrow x = 0, \quad y = \alpha t \quad \text{for some} \quad 0 \leq \alpha \in \mathbf{R}$$

$$\Rightarrow \beta x = 0 \quad \text{and} \quad \beta y = \beta \alpha t \quad \text{for some} \quad 0 \leq \alpha \in \mathbf{R}$$

$$\Rightarrow \beta x \leq_b \beta y$$

Although $0 \leq_b t$,

$$t + 0 = t \not\leq_b t + t.$$

It is clear that the cardinality of the sets

$$\{\leq_a : \leq_a \text{ a partial order on E with condition (a) and without condition (b)}\}$$

$$\{\leq_b : \leq_b \text{ a partial order on E with condition (b) and without condition (a)}\}$$

are at least $dim(E)$, dimension of $E$

## 2. KANTOROVICH EXTENSION THEOREM

As usual the set of all positive elements of a Riesz space $E$ is denoted by $E^+$. Throughout paper $E$ and $F$ stand for arbitrary Archimedean Riesz spaces. Recall that a map $T : E^+ \to F$ is called *additive* if $T(x + y) = T(x) + T(y)$ for all $x, y \in E^+$. A linear map $T : E \to F$ is called *positive* if $T(x) \geq T(y)$ holds whenever $x \geq y$ in $E$. As usual for each $x \in E$ we write

$$x^+ = \sup\{x, 0\} \quad x^- = \sup\{-x, 0\}.$$

For the elementary theory of Riesz space and terminology not explained here we refer to [2].

The starting point in the theory of positive operators is a fundamental extension theorem of L. V. Kantorovich. This theorem can be stated as follows.

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**Theorem 2.1.** *Let $T : E^+ \to F^+$ be an additive map. Then t extends uniquely to a positive operator $T : E \to F$. Moreover, the unique extension $T$ is given by*

$$T(x) = t(x^+) - t(x^-)$$

*for each $x \in E$.*

The proof of this very important theorem can be found in any book on Riesz space theory. For example see Theorem 1.7 in [1].

Recall that a linear map $T : E \to F$ is called *order bounded* if $T([x, y])$ is an order bounded subset of $F$ for each $x, y \in F$ with $x \leq y$. We call also a map $t : E^+ \to F$ is order bounded if $T([x, y])$ is an order bounded subset of $F$ for each $x, y \in F$ with $0 \leq x \leq y$. The Kantorovich theorem can be generalized as follows.

**Theorem 2.2.** *Let $t : E^+ \to F$ be a additive map. If $\sup T([0, x])$ and $\inf T([0, x])$ exists for each $x \in E^+$ then $T$ extends uniquely to an order bounded operator $T : E \to F$.*

*Proof.* Let $t_1, t_2 : E^+ \to F^+$ be defined by

$$t_1(x) = \sup t([0, x]) \quad \text{and} \quad t_2(x) = \inf t([0, x]).$$

Then $t_1$ and $t_2$ are additive. To see this let $x, y \in E^+$ be given. If $0 \leq a \leq x$ and $0 \leq b \leq y$ then $0 \leq a + b \leq x + y$ so $t(a) + t(b) = t(a + b) \leq t_1([0, x + y])$. Since $a$ and $b$ are arbitrary we have $t_1(x) + t_1(y) \leq t_1(x + y)$. Let $0 \leq z \leq x + y$. From the Riesz decomposition theorem we can choose $0 \leq a \leq x$, $0 \leq b \leq y$ with $z = a + b$. Then $t(z) = t(a) + t(b) \leq t_1(x) + t_2(y)$. If we take the supremum of $t(z)$ over $0 \leq z \leq x + y$ we have $t_1(x + y) \leq t_1(x) + t_1(y)$. Hence $t_1$ is additive. Similarly we can show that $t_2$ is additive. From the Kantorovich theorem $t_1$ and $t_2$ have unique positive operator extensions $T_1, T_2 : E \to F$, respectively. Moreover,

$$T_1(x) = t_1(x^+) - t_1(x^-) \quad \text{and} \quad T_2(x) = t_2(x^+) - t_2(x^-)$$

for all $x \in E$. Now we claim that $t(x) = t_1(x) + t_2(x)$ for each $x \in E^+$. Let $0 \leq z \leq x$ in $E$. Then

$$t(z) = t(x) - t(x - z) \leq t(x) - \inf t([0, x]).$$

This implies that $t_1(x) + t_2(x) \leq t(x)$. On the other hand

$$t(x) - t_2(x) = \sup\{t(x) - t(y) : 0 \leq y \leq x\} = \sup\{t(x - y) : 0 \leq y \leq x\} \leq t_1(x).$$

This shows $t_1(x) + t_2(x) = t(x)$. Hence $T = T_1 + T_2$ is a linear extension of $t$. It is clear that $T$ is order bounded. It is clear that $T$ is unique extension. This completes the proof. □

If $t : E^+ \to F^+$ is additive then $\sup t([0, x]) = t(x)$ and $\inf t([0, x]) = t(0) = 0$. This shows that Theorem 2 is an extension of Theorem 1. Let $\mathcal{L}_b(E^+, F)$ be the set of all order bounded additive maps from $E^+$ into $F$. It is clear that $\mathcal{L}_b(E^+, F)$ is an ordered vector space under the pointwise algebraic operations and ordering $S \geq T$ whenever $S(x) \geq T(x)$ holds for all $x \in E^+$. Moreover we have the following corollary.

**Corollary 2.3.** *If $F$ is Dedekind complete then $\mathcal{L}_b(E^+, F)$ is a Dedekind complete Riesz space. Moreover $\mathcal{L}_b(E^+, F)$ is Riesz isomorphic to $\mathcal{L}_b(E, F)$, the space of all order bounded operators from $E$ into $F$.*

*Proof.* Let $\pi : \mathcal{L}_b(E^+, F) \to \mathcal{L}_b(E, F)$ be defined as $\pi(t)$ is the unique order bounded linear extension of $t$. It is obvious that $\pi$ is a Riesz isomorphism. □

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