CREATIVE MATH. & INF.
19 (2010), No. 1, 45 - 52

Online version available at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Statistical approximation by q-integrated Meyer-König-Zeller-Kantorovich operators

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ABSTRACT.

In the present paper we introduce a q-analogue of the integrated Meyer-König-Zeller-Kantorovich type operators and investigate their statistical approximation properties.

1. INTRODUCTION

In the last decade some new generalizations of well known positive linear operators, based on the q-integers were introduced and studied by several authors. For instance q-Meyer-König and Zeller operators were studied by Trif [11], Doğru and Gupta [6], Dogru et al. [4] and Dogru and Duman [5] etc. In 2008 M. Ali Özarslan and Oktay Duman [2] proposed an approximation theorem by Meyer-König and Zeller type operators. C. Radu [10] in 2008 proposed statistical approximation by some linear operators of discrete type. In what follows we mention some basic definitions and notations used in q-calculus, details can be found in [9] and [7].

For any fixed real number $q > 0$, we denote q-integers by $[k]$, $k \in \mathbb{N}$

$$[k] = \begin{cases} 1 + q + q^2 + \ldots + q^{k-1} & \text{if} \quad q \neq 1, \\ k & \text{if} \quad q = 1. \end{cases}$$

We set $[0]_q = 0$. In general, for a real number $k \in \mathbb{R}$, we denote the q-number k by

$$[k] = \begin{cases} \dfrac{1-q^k}{1-q} & \text{if} \quad q \neq 1, \\ k & \text{if} \quad q = 1. \end{cases}$$

The q-factorial is defined as follows

$$[k]! = \begin{cases} [1] \cdot [2] \cdot \ldots \cdot [k] & \text{if} \quad k = 1, 2, \ldots \\ 1 & \text{if} \quad k = 0, \end{cases}$$

and the q-binomial coefficients are given by

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]!\,[n-k]!}, \quad 0 \leq k \leq n.$$

Also

$$\sum_{k=0}^{\infty} \left[\begin{array}{c} n+k-1 \\ k \end{array} \right] x^k = \frac{1}{\prod_{j=0}^{n-1} (1 - q^j x)} \tag{1.1}$$

and

$$\frac{[k+1]}{[n+k+1]} - \frac{[k]}{[n+k]} = \frac{q^k [n]}{[n+k][n+k+1]}. \tag{1.2}$$

The q-analogue of integration (see [3]) is defined as

$$\int_0^a f(t) d_q t = (1-q) a \sum_{j=0}^{\infty} f(q^j a) q^j. \tag{1.3}$$

For $q \in (0,1)$, $x \in [0,1]$ and $n \in \mathbb{N}$, we propose the q-Meyer-König-Zeller-Kantorovich operators as

$$M_{n,q}(f;x) = [n+1] \sum_{k=0}^{\infty} \left[\begin{array}{c} n+k+1 \\ k \end{array} \right] x^k q^{-k} P_{n-1}(x) \int_{[k]/[n+k]}^{[k+1]/[n+k+1]} f(t) d_q t, \tag{1.4}$$

where $P_{n-1}(x) = \prod_{j=0}^{n-1} (1 - q^j x)$.

Remark 1.1. It can be seen that for $q \to 1^-$ the q-Meyer-König-Zeller and Kantorovich operator becomes the operator studied in [1].

Received: 19.03.2009; In revised form: 27.01.2010; Accepted: 08.02.2010.
2000 *Mathematics Subject Classification.* 41A25, 41A35.
Key words and phrases. *q-integers, q-integrated Meyer-König-Zeller-Kantorovich operators, statistical approximation.*

2. Moments

Lemma 2.1. *For $r = 0, 1, 2\dots$ and $n > r$, we have*

$$P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]\frac{x^k}{[n+k-1]^{\underline{r}}} = \frac{\prod_{j=1}^{r}(1-q^{n-j}x)}{[n-1]^{\underline{r}}} \tag{2.5}$$

where $[n-1]^{\underline{r}} = [n-1][n-2]\dots[n-r]$.

Proof. Clearly for $r = 0$, relation holds.
For $r = 1$, using identity (1.1), we get

$$\begin{aligned} P_{n-1}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]\frac{x^k}{[n+k-1]} &= \frac{(1-q^{n-1}x)}{[n-1]}P_{n-2}(x)\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-2 \\ k \end{array}\right]x^k \\ &= \frac{(1-q^{n-1}x)}{[n-1]}. \end{aligned}$$

By method of induction the above lemma can be proved easily. □

Lemma 2.2. *The following inequality holds true*

$$\frac{1}{[n+k+r]} \leq \frac{1}{q^{r+1}[n+k-1]},\ r \geq 0. \tag{2.6}$$

Proof of lemma is very technical and omitted here.

Lemma 2.3. *For all $x \in [0,1]$, $n \in \mathbb{N}$ and $q \in (0,1)$, we have*

$$M_{n,q}(e_0;x) = 1, \tag{2.7}$$

$$M_{n,q}(e_1;x) \leq \frac{1}{([3]-1)}\left(2x + \frac{(1-q^{n-1}x)}{q[n-1]}\right), \tag{2.8}$$

$$\begin{aligned} M_{n,q}(e_2;x) \leq \frac{1}{[3]}\Bigg(\frac{3x^2}{q^2} + \frac{3x}{q^3}\left(1+\frac{1}{q}\right)\frac{(1-q^{n-1}x)}{[n-1]^{\underline{1}}} \\ + \frac{1}{q^5}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1]^{\underline{2}}}\Bigg). \end{aligned} \tag{2.9}$$

Proof. In (1.4), by using (1.1), (1.2) and (1.3), we have

$$\begin{aligned} M_{n,q}(e_0;x) &= [n+1]\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k+1 \\ k \end{array}\right]x^kq^{-k}P_{n-1}(x)\int_{[k]/[n+k]}^{[k+1]/[n+k+1]}d_qt \\ &= [n+1]\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k+1 \\ k \end{array}\right]x^kq^{-k}P_{n-1}(x)\left(\frac{[k+1]}{[n+k+1]}-\frac{[k]}{[n+k]}\right) \\ &= [n+1]\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k+1 \\ k \end{array}\right]x^kq^{-k}P_{n-1}(x)\left(\frac{q^k[n]}{[n+k][n+k+1]}\right) \\ &= \sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right]x^kP_{n-1}(x) \\ &= 1. \end{aligned}$$

Using Lemma 2.1, Lemma 2.2 and the identities (1.1) and (1.2), we obtain the relation (2.8) as follows

$$\begin{aligned} \int_{[k]/[n+k]}^{[k+1]/[n+k+1]} td_qt &= (1-q)\left(\frac{[k+1]^2}{[n+k+1]^2}\sum_{k=0}^{\infty}q^{2j} - \frac{[k]^2}{[n+k]^2}\sum_{k=0}^{\infty}q^{2j}\right) \\ &= \frac{1}{(1+q)}\left(\frac{[k+1]^2}{[n+k+1]^2}-\frac{[k]^2}{[n+k]^2}\right) \\ &= \frac{1}{(1+q)}\frac{q^k[n]}{[n+k][n+k+1]}\left(\frac{q[k]+1}{[n+k+1]}+\frac{[k]}{[n+k]}\right) \\ &= \frac{1}{(1+q)}\frac{q^k[n]}{[n+k][n+k+1]}\left([k]\left(\frac{q}{[n+k+1]}+\frac{1}{[n+k]}\right)+\frac{1}{[n+k+1]}\right) \\ &\leq \frac{1}{(1+q)}\frac{q^k[n]}{[n+k][n+k+1]}\left(\frac{[k]}{q}\frac{2}{[n+k-1]}+\frac{1}{q^2[n+k-1]}\right). \end{aligned}$$

$$M_{n,q}(e_1,x) \le \frac{[n+1]}{(1+q)}\sum_{k=0}^{\infty}\begin{bmatrix} n+k+1 \\ k \end{bmatrix} x^k q^{-k}$$
$$\left(\frac{q^k[n]}{[n+k][n+k+1]}\left(\frac{[k]}{q}\frac{2}{[n+k-1]}+\frac{1}{q^2[n+k-1]}\right)\right) = \frac{1}{([3]-1)}$$
$$\left(2\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{[k]}{[n+k-1]}x^k P_{n-1}(x)+\frac{1}{q}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{x^k}{[n+k-1]}P_{n-1}(x)\right)$$
$$= \frac{1}{([3]-1)}\left(2\sum_{k=1}^{\infty}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix} x^k P_{n-1}(x)+\frac{1}{q}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{x^k}{[n+k-1]}P_{n-1}(x)\right)$$
$$= \frac{1}{([3]-1)}\left(2x\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k P_{n-1}(x)+\frac{1}{q}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix}\frac{x^k}{[n+k-1]}P_{n-1}(x)\right)$$
$$= \frac{1}{([3]-1)}\left(2x+\frac{(1-q^{n-1}x)}{q[n-1]}\right).$$

A similar calculation reveals relation (2.9) as follows:

$$\int_{[k]/[n+k]}^{[k+1]/[n+k+1]} t^2 d_q t = \frac{(1-q)}{(1-q^3)}\left(\frac{[k+1]^3}{[n+k+1]^3}-\frac{[k]^3}{[n+k]^3}\right)$$
$$= \frac{1}{[3]}\frac{q^k[n]}{[n+k][n+k+1]}\left([k]^2 b_2(n,k)+[k]b_1(n,k)+b_0(n,k)\right)$$

where

$$b_2(n,k) = \frac{q^2}{[n+k+1]^2}+\frac{1}{[n+k]^2}+\frac{q}{[n+k+1][n+k]}$$
$$b_1(n,k) = \frac{2q}{[n+k+1]^2}+\frac{1}{[n+k+1][n+k]}$$
$$b_0(n,k) = \frac{1}{[n+k+1]^2},$$

$$M_{n,q}(e_2,x) = \frac{1}{[3]}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k\left([k]^2 b_2(n,k)+[k]b_1(n,k)+b_0(n,k)\right).$$

Using Lemma 2.2, we get

$$b_2(n,k) \le \frac{3}{q^3[n+k-1]^{\underline{2}}}$$
$$b_1(n,k) \le \frac{3}{q^4[n+k-1]^{\underline{2}}}$$
$$b_0(n,k) \le \frac{1}{q^5[n+k-1]^{\underline{2}}}$$

$$\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k [k]^2 b_2(n,k) P_{n-1}(x) \le \frac{3}{q^3}\sum_{k=1}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^k \frac{[k]^2}{[n+k-1]^{\underline{2}}}P_{n-1}(x)$$
$$= \frac{3}{q^3}\sum_{k=1}^{\infty}\begin{bmatrix} n+k-2 \\ k-1 \end{bmatrix} x^k \frac{[k]}{[n+k-2]}P_{n-1}(x)$$
$$= \frac{3}{q^3}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^{k+1} \frac{[k+1]}{[n+k-1]}P_{n-1}(x)$$
$$= \frac{3}{q^3}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^{k+1} \frac{1}{[n+k-1]}P_{n-1}(x)$$
$$+\frac{3}{q^2}\sum_{k=0}^{\infty}\begin{bmatrix} n+k-1 \\ k \end{bmatrix} x^{k+2} P_{n-1}(x)$$
$$= \frac{3x^2}{q^2}+\frac{3x}{q^3}\frac{(1-q^{n-1}x)}{[n-1]}$$

and

$$\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right] x^k [k] b_1(n,k) P_{n-1}(x) \leq$$

$$\frac{3}{q^4}\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right] x^k \frac{[k]}{[n+k-1][n+k-2]} P_{n-1}(x)$$

$$\leq \frac{3}{q^4}\sum_{k=1}^{\infty}\left[\begin{array}{c} n+k-2 \\ k-1 \end{array}\right] x^k \frac{1}{[n+k-2]} P_{n-1}(x) = \frac{3x}{q^4}\frac{(1-q^{n-1}x)}{[n-1]}.$$

Also

$$\sum_{k=0}^{\infty}\left[\begin{array}{c} n+k-1 \\ k \end{array}\right] x^k b_0(n,k) P_{n-1}(x) \leq \frac{1}{q^5}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1]\underline{2}},$$

therefore

$$M_{n,q}(e_2,x) \leq$$

$$\frac{1}{[3]}\left(\frac{3x^2}{q^2} + \frac{3x}{q^3}\frac{(1-q^{n-1}x)}{[n-1]} + \frac{3x}{q^4}\frac{(1-q^{n-1}x)}{[n-1]} + \frac{1}{q^5}\frac{(1-q^{n-1}x)(1-q^{n-2}x)}{[n-1]\underline{2}}\right).$$

□

3. STATISTICAL APPROXIMATION PROPERTIES

In this section, by using a Bohman-Korovkin type theorem proved in [8], we present the statistical approximation properties of the operator $M_{n,q}$ given by (1.4).

At this moment, we recall the concept of statistical convergence.

A sequence $(x_n)_n$ is said to be statistically convergent to a number L, denoted by $st - \lim_n x_n = L$ if, for every $\varepsilon > 0$,

$$\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0, \tag{3.10}$$

where

$$\delta(S) := \frac{1}{N}\sum_{k=1}^{N} \chi S(j)$$

is the natural density of set $S \subseteq \mathbb{N}$ and χS is the characteristic function of S.

Let $C_B(D)$ represent the space of all continuous functions on D and bounded on entire real line, where D is any interval on real line. It can be easily shown that $C_B(D)$ is a Banach space with supreme norm. Also $M_{n.q}(f,x), n \in \mathbb{N}$ are well defined for any $f \in C_B([0,1])$.

Theorem A. [5]. *Let $(L_n)_n$ be a sequence of positive linear operators from $C_B([a,b])$ into $B([a,b])$ and satisfies the condition that*

$$st - \lim_n \|L_n e_i - e_i\| = 0 \text{ for all } i = 0,1,2.$$

Then

$$st - \lim_n \|L_n f - f\| = 0 \text{ for all } f \in C_B([a,b]).$$

We consider a sequence $(q_n)_n$, $q_n \in (0,1)$, such that

$$st - \lim_n q_n = 1. \tag{3.11}$$

As an application of Theorem A, we have the following result for our operators.

Theorem 3.1. *Let $(q_n)_n$ be a sequence satisfying (3.11). Then for the operators $M_{n,q}f$ satisfying the condition*

$$st - \lim_n \|M_{n,q} e_i - e_i\| = 0 \text{ for all } i = 0,1,2$$

we have

$$st - \lim_n \|M_{n,q} f - f\| = 0 \text{ for all } f \in C_B([a,b]).$$

Proof. It is clear that

$$st - \lim_n \|M_{n,q_n}(e_0;\cdot) - e_0\| = 0. \tag{3.12}$$

Based on Lemma 2.3, we have

$$\begin{aligned}|M_{n,q_n}(e_1;\cdot)-e_1| &\le \left|\left(\frac{2}{[3]_{q_n}-1}-1\right)x+\frac{(1-q_n^{n-1}x)}{([3]_{q_n}-1)q_n[n-1]_{q_n}}\right| \\ &\le \left|\frac{3-[3]_{q_n}}{[3]_{q_n}-1}\right|+\frac{1}{|([3]_{q_n}-1)q_n[n-1]_{q_n}|}+|\frac{q_n^{n-2}}{([3]_{q_n}-1)[n-1]_{q_n}|} \\ &\le |3-[3]_{q_n}|+\frac{1}{|q_n[n-1]_{q_n}|}+\frac{q_n^{n-2}}{|[n-1]_{q_n}|}.\end{aligned}$$

Since $st-\lim_n q_n=1$, we get

$$st-\lim_n \frac{1}{[n-1]_{q_n}}=0 \tag{3.13}$$

and

$$st-\lim_n(|3-[3]_{q_n}|)=0. \tag{3.14}$$

Hence, we get

$$\|M_{n,q_n}(e_1;\cdot)-e_1\|<\epsilon. \tag{3.15}$$

Define the following sets

$$\begin{aligned}A&:=\{n\in\mathbb{N}:\|M_{n,q_n}(e_1;\cdot)-e_1\|\ge\epsilon\},\\ A_1&:=\{n\in\mathbb{N}:(3-[3]_{q_n})\ge\epsilon/3\},\\ A_2&:=\{n\in\mathbb{N}:\frac{1}{[n-1]_{q_n}}\ge\epsilon/3\}.\end{aligned}$$

Thus we obtain $A\subseteq A_1\bigcup A_2$ i.e. $\delta(A)\le\delta(A_1)+\delta(A_2)=0$.
Hence

$$st-\lim_n\|M_{n,q_n}(e_1;\cdot)-e_1\|=0. \tag{3.16}$$

A similar calculation reveals

$$\begin{aligned}|M_{n,q_n}(e_2,\cdot)-e_2| &\le \frac{1}{q^5}\left(\frac{|3-[3]_{q_n}|}{[3]_{q_n}}+\frac{6}{[n-1]_{q_n}}+\frac{q_n^{n-1}}{[n-1]_{q_n}}+\frac{(1-q_n^{n-1})^2}{[n-2]_{q_n}^2}\right)\\ &\le \frac{1}{q^5}|3-[3]_{q_n}|+\frac{1}{q^5}\frac{6}{[n-1]_{q_n}}+\frac{1}{q^5}\frac{q_n^{n-1}}{[n-1]_{q_n}}+\frac{1}{q^5}\frac{1}{[n-2]_{q_n}^2}+\frac{q_n^{2n-7}+2q_n^{n-6}}{[n-2]_{q_n}^2}.\end{aligned}$$

By using (3.13) and (3.14) we obtain $\|M_{n,q_n}(e_2,\cdot)-e_2\|<\epsilon$.
Define the following sets

$$\begin{aligned}B&=\{n\in\mathbb{N}:\|M_{n,q_n}(e_2;\cdot)-e_2\|\ge\epsilon\},\\ B_1&:=\{n\in\mathbb{N}:(3-[3]_{q_n})\ge\epsilon/3\},\\ B_2&:=\left\{n\in\mathbb{N}:\frac{1}{[n-1]_{q_n}}\ge\epsilon/36\right\},\\ B_3&:=\left\{n\in\mathbb{N}:\frac{1}{[n-2]_{q_n}}\ge\epsilon/18\right\}.\end{aligned}$$

Thus we obtain $B\subseteq B_1\bigcup B_2\bigcup B_3$ i.e. $\delta(B)\le\delta(B_1)+\delta(B_2)+\delta(B_3)=0$.
Hence

$$st-\lim_n\|M_{n,q_n}(e_2;\cdot)-e_2\|=0. \tag{3.17}$$

Thus by using (3.11), (3.16), (3.17) and Theorem A, result holds. This completes the proof of Theorem 3.1. □

Corollary 3.1. *Let $(q_n)_n$ be a sequence satisfying $\lim_n q_n=1$, then for all $f\in C_B([0,1])$ we have*

$$\lim_n\|M_{n,q}(f;\cdot)-f\|=0.$$

Follows by just replacing statistical convergence with uniform convergence.

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