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# On the generalized Stirling formula

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ABSTRACT.

The aim of this paper is to give a numerical construction of the best approximations of the factorial function among the family of approximations introduced by Mortici [Arch. Math. (Basel) 93 (2009), No. 1, 37-45].

## 1. INTRODUCTION

There are many situations when for solving some practical problems, big factorials must be estimated. Probably the most known estimation formula for the factorial function is the Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1.1}$$

which will be written for some reasons in the equivalent form:

$$n! \sim \sqrt{2\pi e}\left(\frac{n}{e}\right)^{n+\frac{1}{2}} = \sigma_n. \tag{1.2}$$

The formula (1.1) was discovered by the French mathematician Abraham de Moivre (1667-1754) in the form

$$n! \sim k\left(\frac{n}{e}\right)^n \sqrt{n}$$

(where the constant was not explicitly given) when he was preoccupied to give the normal approximation for the binomial distribution.

Afterwards, the Scottish mathematician James Stirling (1692-1770) discovered the constant $\sqrt{2\pi}$, using the Wallis' formula. Both used a formula of type Euler-MacLaurin to estimate the sum $\ln 2 + \ln 3 + ... + \ln n$. For proofs and details see for example [13]-[15].

A slightly more accurate estimate is the Burnside formula [1]:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \beta_n \tag{1.3}$$

and other better formulas were found [2], but a sacrifice of simplicity.

Recently, Mortici [3] introduced the formula

$$n! \sim \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} = \alpha_n$$

and the more general family of approximations

$$n! \sim \lambda\left(\frac{n+a}{e}\right)^{n+b} = \mu_n(\lambda, a, b), \tag{1.4}$$

where the constants $\lambda$, $a$, $b$ will be determined to obtain performant results. Please see also [3]-[12] for other details. Interesting fact is that also the Stirling's formula is a particular case of (1.4), obtained for $\lambda = \sqrt{2\pi e}$, $a = 0$ and $b = 1/2$, as we can see from the representation (1.2).

In order to have approximations of type (1.4), the basic condition that the sequence $(\mu_n(\lambda, a, b)/n!)_{n\geq 1}$ converges to 1 must be fulled. It is obtained in [3] the following final form of approximations, for $x \in [0, 1]$,

$$n! \sim \sqrt{2\pi e} \cdot e^{-x}\left(\frac{n+x}{e}\right)^{n+\frac{1}{2}} = \mu_n(x),$$

and then the best constants $x \in \{\omega, \zeta\}$, $\omega = (3-\sqrt{3})/6$ and $\zeta = (3+\sqrt{3})/6$ are deduced to provide the best approximations. These constants were found considering some completely monotonic functions.

In this paper we refind these constants using a different numerical method.

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## 2. The Results

The function $\mu_n(x)$ is strictly increasing on $[0, 1/2]$ and strictly decreasing on $[1/2, 1]$, because

$$\frac{d}{dx}(\ln \mu_n(x)) = \frac{\frac{1}{2} - x}{x + n},$$

so a direct consequence is that $\mu_n(0) < n! < \mu_n(1/2) > n! > \mu_n(1)$, or

$$\sigma_n < n! < \beta_n > n! > \alpha_n. \tag{2.5}$$

By the inequalities (2.5) and by continuity arguments on the function $\mu_n(x)$, it results that there exists a sequence $(w_n)_{n\geq 1} \subset (0, 1/2)$ such that $\mu_n(w_n) = n!$, which is

$$n! = \sqrt{2\pi e} \cdot e^{-w_n} \left(\frac{n + w_n}{e}\right)^{n+\frac{1}{2}}$$

and there exists a sequence $(z_n)_{n\geq 1} \subset (1/2, 1)$ such that $f(z_n) = n!$, or

$$n! = \sqrt{2\pi e} \cdot e^{-z_n} \left(\frac{n + z_n}{e}\right)^{n+\frac{1}{2}}.$$

After some numerical computations, we deduced that the sequences $(w_n)_{n\geq 1}$ and $(z_n)_{n\geq 1}$ are decreasing, so they are convergent to $w = 0.21133...$, respective to $z = 0.78868...$ . Now we have obtained the following under-approximation and upper-approximation formulas for the factorial function:

$$\sqrt{2\pi e} \cdot e^{-0.21133...} \left(\frac{n + 0.21133...}{e}\right)^{n+\frac{1}{2}} < n! < \sqrt{2\pi e} \cdot e^{-0.78868...} \left(\frac{n + 0.78868...}{e}\right)^{n+\frac{1}{2}}.$$

Further, we remark that $w + z \approx 1$ and $wz \approx 1/6$, so we take $w = (3 - \sqrt{3})/6$ and $z = (3 + \sqrt{3})/6$ to obtain the approximations:

$$\omega_n = \sqrt{2\pi} \cdot e^{\frac{\sqrt{3}}{6}} \left(\frac{n + \frac{3-\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}} < n! < \sqrt{2\pi} \cdot e^{\frac{-\sqrt{3}}{6}} \left(\frac{n + \frac{3+\sqrt{3}}{6}}{e}\right)^{n+\frac{1}{2}} = \zeta_n. \tag{2.6}$$

## 3. Concluding Remarks

In the final part, we expose the idea for obtaining new, stronger approximations. To be more exactly, if we replace the approximations (2.6) by their geometric mean, then we obtain the following much more accurate estimation:

$$n! \approx \sqrt{2\pi} \left(\frac{n^2 + n + \frac{1}{6}}{e^2}\right)^{\frac{n}{2}+\frac{1}{4}} = \mu_n. \tag{3.7}$$

One way to compare the accurateness of some approximations is to introduce the number of exact decimal digits function (edd), by the formula

$$edd(n) = -\lg\left|1 - \frac{approx(n)}{n!}\right|,$$

where $approx(n)$ is the respective approximation. We take into account the approximation formulas $\alpha_n$ from (2.5), $\omega_n$ and $\zeta_n$ from (2.6), the Stirling's formula $\sigma_n$ from (1.2), the Burnside's formula $\beta_n$ from (1.3) and the following formula from [2] due to R. W. Gosper:

$$\gamma_n = \sqrt{\left(2n + \frac{1}{3}\right)\pi} \cdot \left(\frac{n}{e}\right)^n.$$

As we can see, the most accurate is our formula $\mu_n$ from (3.7), while only the Gosper's formula provides comparable results with our new formulas $\omega_n$ and $\zeta_n$.

| $n$ | $edd(n)$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\sigma_n$ | $\alpha_n$ | $\beta_n$ | $\omega_n$ | $\zeta_n$ | $\gamma_n$ | $\mu_n$ |
| 50 | 6.3978 | 6.4176 | 7.0996 | 12.660 | 12.680 | 12.804 | **17.247** |
| 100 | 7.0905 | 7.1004 | 7.7880 | 14.041 | 14.051 | 14.185 | **19.311** |
| 250 | 8.0065 | 8.0105 | 8.7014 | 15.871 | 15.875 | 16.015 | **22.051** |
| 1000 | 9.3927 | 9.3937 | 10.086 | 18.642 | 18.643 | 18.786 | **26.205** |

Finally, note that if the Stirling's formula gives $1000!$ with the first $9$ exact decimals and Gosper's formula gives the same result with $18$ exact decimals, then our formula $\mu_n$ provides the value of $1000!$ with $26$ exact decimals.

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