CREATIVE MATH. & INF.
19 (2010), No. 1, 76 - 84

Online version available at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Subsets defined in terms of envelopes and weak envelopes

V. RENUKADEVI

ABSTRACT.
We define and discuss the various characterizations and properties of some kind of sets in monotonic spaces, weak envelope spaces and envelope spaces which are similar to that of dense sets and nowheredense sets in topological spaces.

1. INTRODUCTION AND PRELIMINARIES

We first give the necessary definitions of $Császár$ [1]. Let X be a nonempty set and $\gamma : \wp(X) \to \wp(X)$. We say that $\gamma \in \Gamma(X)$ or simply $\gamma \in \Gamma$ if $\gamma(A) \subset \gamma(B)$ whenever $A \subset B$ where A and B are subsets of X. If $\gamma \in \Gamma$, we call the pair (X, γ), a *monotonic space*. A subset A of X is $\gamma-$open [1] if $A \subset \gamma(A)$. The complement of a $\gamma-$open set is said to be a $\gamma-$closed set. If $\gamma \in \Gamma$, then $\gamma^\star : \wp(X) \to \wp(X)$ is defined by $\gamma^\star(A) = X - \gamma(X - A)$ and $\gamma^\star \in \Gamma$ [1]. A subset A of X is $\gamma^\star-$closed if and only if $\gamma(A) \subset A$ [1, Proposition 1.8]. Let $\xi = \{A \subset X \mid A = \gamma(A)\}$. ξ is called the family of all $\gamma-$regularclosed ($\gamma-$regular [1])sets. Therefore, a subset A of X is $\gamma-$regularclosed if and only if A is $\gamma-$open and $\gamma^\star-$closed. The complement of a $\gamma-$regularclosed set is called a $\gamma-$regularopen set. Let μ be the family of all $\gamma-$regularopen sets. Then, $A \in \mu$ if and only if $A = \gamma^\star(A)$ if and only if A is $\gamma-$closed and $\gamma^\star-$open. We have the following subclasses of Γ.
$\Gamma_0 = \{\gamma \in \Gamma \mid \gamma(\emptyset) = \emptyset\}$,
$\Gamma_1 = \{\gamma \in \Gamma \mid \gamma(X) = X\}$,
$\Gamma_2 = \{\gamma \in \Gamma \mid \gamma(\gamma(A)) = \gamma(A)$ for every subset A of $X\}$,
$\Gamma_- = \{\gamma \in \Gamma \mid \gamma(A) \subset A$ for every subset A of $X\}$ and
$\Gamma_+ = \{\gamma \in \Gamma \mid A \subset \gamma(A)$ for every subset A of $X\}$. If $I = \{0, 1, 2, +, -\}$ and $A \subset I$, then $\gamma \in \Gamma_A$ if and only if $\gamma \in \Gamma_i$ for every $i \in A$. If $\gamma \in \Gamma_+$, then γ is called a *weak envelope* [3]. The pair (X, γ) is called a *weak envelope* space. If $\gamma \in \Gamma_{2+}$, then γ is called an *envelope* [3]. The pair (X, γ) is called an *envelope* space. Weak envelope and envelope operations are further studied by $Á.\ Császár$, in [4].

A subset μ of $\wp(X)$ is called a *generalized topology* (briefly GT) [2] if $\emptyset \in \mu$ and arbitrary union of members of μ is again in μ. Elements of μ are called *$\mu-$open* sets. Complements of $\mu-$open sets are called *$\mu-$closed* sets. In this paper, we define and discuss the various characterizations and properties of sets in monotonic spaces, weak envelope spaces and envelope spaces which are similar to that of dense sets and nowheredense sets in topological spaces.

2. RC-DENSE AND RC-NWDENSE SETS

Let (X, γ) be a monotonic space. A subset A of X is said to be *rc-dense* if $\gamma(A) = X$. It is clear that $\gamma(A) = X$ if and only if $\gamma^*(X - A) = \emptyset$. Since $\gamma \in \Gamma$, it follows that every superset of a rc-dense set is rc-dense and so the existence of a rc-dense set implies that $\gamma(X) = X$ and so $\gamma \in \Gamma_1$ which says that, by Proposition 1.7 (b) of [1], $\gamma^* \in \Gamma_0$. Equivalently, if $\gamma^* \notin \Gamma_0$, then no rc-dense sets exist. The following Theorem 2.1 gives a property of rc-dense sets.

Theorem 2.1. *Let (X, γ) be a monotonic space. If a subset A of X is rc-dense, then $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V.*

Proof. Suppose $A \cap V = \emptyset$ for some nonempty $\gamma-$regularopen set. $A \cap V = \emptyset$ implies that $V \subset X - A$ and so $V \subset \gamma^*(X - A)$. Therefore, $X - \gamma^*(X - A) \subset X - V$ and so $X = \gamma(A) \subset X - V$ which implies that $V = \emptyset$, a contradiction to the hypothesis. Therefore, $A \cap V \neq \emptyset$ for every nonempty γ-regularopen set V. □

The following Example 2.1 shows that the converse of Theorem 2.1 is not true.

Example 2.1. *Let $X = \{a, b, c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{b\}, \gamma(\{a\}) = \{a, b\}, \gamma(\{b\}) = \{b, c\}, \gamma(\{c\}) = \{b, c\}, \gamma(\{a, b\}) = X, \gamma(\{a, c\}) = X, \gamma(\{b, c\}) = \{b, c\}, \gamma(X) = X$. Then $\gamma \in \Gamma$, $\gamma \notin \Gamma_2$, $\xi = \{\{b, c\},\ X\}$ and $\mu = \{\emptyset, \{a\}\}$. Now $V = \{a\}$ is the only nonempty $\gamma-$regularopen set such that $V \cap \{a\} \neq \emptyset$ but $\{a\}$ is not rc-dense. Note that $\{a, b\}$, $\{a, c\}$ and X are rc-dense sets.*

Theorem 2.2. *Let (X, γ) be a weak envelope space. If A is a subset of X such that $A \cap V \neq \emptyset$ for every nonempty $\gamma-$regularopen set V, then $\gamma(A) \cap V \neq \emptyset$ for every non empty $\gamma-$regularopen set V.*

Proof. The proof follows from the fact that $\gamma \in \Gamma_+$. □

The following Example 2.2 shows that the condition $\gamma \in \Gamma_+$ in Theorem 2.2 cannot be dropped.

Received: 20.10.2008; In revised form: 05.01.2010; Accepted: 08.02.2010.
2000 *Mathematics Subject Classification.* 54A05.
Key words and phrases. *Weak envelope, envelope, generalized topology, quasi-topology, $\mu-$friendly operation, ideal.*

Example 2.2. Let $X = \{a, b, c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{a\}$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{a\}$, $\gamma(\{c\}) = X$, $\gamma(\{a, b\}) = \{a\}$, $\gamma(\{a, c\}) = X$, $\gamma(\{b, c\}) = X$, $\gamma(X) = X$. Then $\gamma \in \Gamma$, $\gamma \notin \Gamma_+$, $\xi = \{\{a\}, X\}$ and $\mu = \{\emptyset, \{b, c\}\}$. Let $A = \{a, b\}$. $V = \{b, c\}$ is the only nonempty $\gamma-$regularopen set such that $V \cap A = \{b\} \neq \emptyset$. But $\gamma(A) = \{a\}$ and so $\gamma(A) \cap V = \emptyset$.

Theorem 2.3. *Let (X, γ) be a monotonic space and $A \subset X$. Then the following hold.*
(a) *If A is rc-dense, then $\gamma(A) \cap V \neq \emptyset$ for every nonempty $\gamma-$regularopen set V.*
(b) *If $\gamma \in \Gamma_2$ and $\gamma(A) \cap V \neq \emptyset$ for every nonempty $\gamma-$regularopen set V, then A is rc-dense.*

Proof. (a) The proof of (a) is clear.
(b) Suppose $\gamma(A) \cap V \neq \emptyset$ for every nonempty $\gamma-$regularopen set V and A is not rc-dense. Then $X - \gamma(A) \neq \emptyset$. If $V = X - \gamma(A)$, then $\gamma^\star(V) = \gamma^\star(X - \gamma(A)) = X - \gamma(X - (X - \gamma(A))) = X - \gamma(\gamma(A)) = X - \gamma(A) = V$ and so V is $\gamma-$regularopen. But $V \cap \gamma(A) = (X - \gamma(A)) \cap \gamma(A) = \emptyset$, a contradiction to the hypothesis. Therefore, A is rc-dense. □

The following Example 2.3 shows that the condition $\gamma \in \Gamma_2$ in the above Theorem 2.3 (b) cannot be dropped. The proof of the Corollary 2.1 follows from Theorems 2.1, 2.2 and 2.3.

Example 2.3. Let (X, γ) be the monotonic space of Example 2.1. $\gamma \notin \Gamma_2$, $V = \{a\}$ is the only nonempty $\gamma-$regularopen set such that $V \cap \gamma(\{a\}) \neq \emptyset$ but $\{a\}$ is not rc-dense.

Corollary 2.1. *Let (X, γ) be an envelope space and $A \subset X$. Then the following are equivalent.*
(a) *A is rc-dense.*
(b) *$A \cap V \neq \emptyset$ for every nonempty $\gamma-$regularopen set V.*
(c) *$\gamma(A) \cap V \neq \emptyset$ for every nonempty $\gamma-$regularopen set V.*

Let (X, γ) be a monotonic space. We say that a subset A of X is said to be *rc-nowheredense* (in short, *rc-nwdense*) if $\gamma^\star\gamma(A) = \emptyset$. It is clear that $\gamma^\star\gamma(A) = \emptyset$ if and only if $\gamma\gamma^\star(X - A) = X$. We will denote the family of all rc-nwdense sets in a monotonic space (X, γ) by $\mathcal{N}$. Since $\gamma \in \Gamma$, it follows that every subset of an rc-nwdense set is an rc-nwdense set and so the existence of an rc-nwdense set implies that $\emptyset$ is rc-nwdense and $\gamma^\star\gamma \in \Gamma_0$. In other words, if $\gamma^\star\gamma \notin \Gamma_0$, then rc-nwdense sets will not exist and so $\mathcal{N} = \emptyset$. The following Example 2.4 shows that in monotonic spaces, we can have either $\mathcal{N} = \emptyset$ or $\mathcal{N} = \{\emptyset\}$ or $\mathcal{N}$ has more than one element. Theorem 2.4 gives a property of rc-nwdense sets.

Example 2.4. (a) [1, Example 1.12] Let $X = \mathbf{R}$ be the set of all real numbers and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(A) = \{0\}$ if $0 \in A$ and $\emptyset$, if otherwise. In this space, $\mathcal{N} = \emptyset$.
(b) Consider the monotonic space of Example 2.1. In this space, $\mathcal{N} = \{\emptyset\}$.
(c) Consider the monotonic space of Example 2.2.
In this space, $\mathcal{N} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

Theorem 2.4. *Let (X, γ) be a monotonic space and $A \subset X$ be rc-nwdense. If V is a nonempty $\gamma-$regularopen set, then V is not a subset of $\gamma(A)$.*

Proof. Since A is rc-nwdense, $\gamma^\star\gamma(A) = \emptyset$ and so $\gamma\gamma^\star(X - A) = X$ which implies that $\gamma^\star(X - A) = X - \gamma(A)$ is rc-dense. By Theorem 2.1, $V \cap (X - \gamma(A)) \neq \emptyset$ for every nonempty $\gamma-$regularopen set V. Therefore, $V - \gamma(A) \neq \emptyset$ which implies that $V \not\subset \gamma(A)$ for every nonempty $\gamma-$regularopen set V. This completes the proof. □

The following Example 2.5 shows that the converse of the above Theorem 2.4 is not true even if the space is a weak envelope space. Theorem 2.5 shows that the converse is true if (X, γ) is an envelope space. Example 2.6 shows that either the condition $\gamma \in \Gamma_2$ or the the condition $\gamma \in \Gamma_+$ cannot be dropped from Theorem 2.5.

Example 2.5. Let (X, γ) be the monotonic space of Example 2.2. If $A = \{b, c\}$, then $V = \{a\}$ is the only nonempty $\gamma-$regularopen set such that $V \not\subset \gamma(A)$. But $\gamma^\star\gamma(A) = \gamma^\star(\{b, c\}) = \{c\} \neq \emptyset$ and so A is not rc-nwdense.

Theorem 2.5. *Let (X, γ) be an envelope space and $A \subset X$. If for every nonempty $\gamma-$regularopen set V, V is not a subset of $\gamma(A)$, then A is rc-nwdense.*

Proof. If V is a nonempty $\gamma-$regularopen set such that V is not a subset of $\gamma(A)$, then $V - \gamma(A) \neq \emptyset$ which implies that $V \cap (X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_+$, by Theorem 2.2, $V \cap \gamma(X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_2$, by Theorem 2.3, $X - \gamma(A)$ is rc-dense and so $\gamma(X - \gamma(A)) = X$ which implies that $X - \gamma^\star\gamma(A) = X$. Therefore, $\gamma^\star\gamma(A) = \emptyset$ and so A is rc-nwdense. □

Example 2.6. (a) Example 2.5 shows that the condition Γ_2 cannot be dropped in Theorem 2.5.
(b) Consider the monotonic space of Example 2.4 (a). Then $\mu = \{\mathbf{R}, \mathbf{R} - \{0\}\}$. Clearly, $\gamma \in \Gamma_2$. If $B \subset X$ such that $0 \in B$ and B has more than one point, then $\gamma(B) = \{0\} \not\supset B$ and so $\gamma \notin \Gamma_+$. If A is a nonempty subset of $\mathbf{R}$ not containing 0, then $\mathbf{R} \not\subset \gamma(A)$. But A is not rc-nwdense.

Let (X, γ) be a monotonic space. A subset A of X is said to be *weak rc-nwdense* (in short, *wrc-nwdense*) if for every nonempty $V \in \mu$, there exists a nonempty $W \in \mu$ with $W \subset V$ such that $W \cap A = \emptyset$. The following Examples 2.7 and 2.8 shows that rc-nwdenseness and wrc-nwdenseness are independent concepts.

Example 2.7. Consider the monotonic space of Example 2.4 (a). Then $\mu = \{\mathbf{R}, \mathbf{R} - \{0\}\}$ and $\{0\}$ is wrc-nwdense but not rc-nwdense. Therefore, a wrc-nwdense set need not be an rc-nwdense set.

Example 2.8. Let $X = \{a, b, c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{a, b\}$, $\gamma(\{c\}) = X$, $\gamma(\{a, b\}) = X$, $\gamma(\{a, c\}) = X$, $\gamma(\{b, c\}) = X$, $\gamma(X) = X$. Then $\mu = \{\emptyset, X, \{b, c\}\}$. If $A = \{b\}$, then A is rc-nwdense but not wrc-nwdense.

Let (X, γ) be a monotonic space. We say that γ is *subadditive* if $\gamma(A \cup B) \subset \gamma(A) \cup \gamma(B)$ for every subsets A and B of X. Since γ is monotonic, if γ is subadditive, then γ is additive. That is, $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$ for every subsets A and B of X. The following Lemma 2.1 is essential to characterize rc-nwdense sets in Theorem 2.6 below.

Lemma 2.1. *Let (X, γ) be a monotonic space and $A \subset X$.*
(a) *$\gamma \in \Gamma_2$ if and only if $\gamma(A)$ is $\gamma-$regularclosed for every subset A of X.*
(b) *If γ is subadditive, then the intersection of two $\gamma-$regularopen sets is a $\gamma-$regularopen set.*
(c) *If $G \cap A = \emptyset$, then $G \cap \gamma(A) = \emptyset$ for every nonempty $\gamma-$regularopen set G. The reverse direction is true, if $\gamma \in \Gamma_+$.*
(d) *If $x \in \gamma(A)$, then $G \cap A \neq \emptyset$ for every $\gamma-$regularopen set G containing x.*
(e) *If $\gamma \in \Gamma_{2+}$ and $G \cap A \neq \emptyset$ for every $\gamma-$regularopen set G containing x, then $x \in \gamma(A)$.*
(f) *A is rc-nwdense if and only if $X - \gamma(A)$ is rc-dense.*

Proof. (a) The proof is clear.
(b) Let U and V be $\gamma-$regularopen. Now $\gamma^\star(U \cap V) = X - \gamma(X - (U \cap V)) = X - \gamma((X - U) \cup (X - V)) = X - (\gamma(X - U) \cup \gamma(X - V)) = (X - \gamma(X - U)) \cap (X - \gamma(X - V)) = \gamma^\star(U) \cap \gamma^\star(V) = U \cap V$ and so $U \cap V$ is $\gamma-$regularopen.
(c) If $G \cap A = \emptyset$, then $A \subset X - G$ and so $\gamma(A) \subset \gamma(X - G) = X - \gamma^\star(G) = X - G$. Therefore, $G \cap \gamma(A) = \emptyset$. The proof of the converse is clear.
(d) If G is a $\gamma-$regularopen set containing x, then $\gamma(A) \cap G \neq \emptyset$. By (c), $G \cap A \neq \emptyset$.
(e) Suppose $x \notin \gamma(A)$. Since $\gamma \in \Gamma_2$, $G = X - \gamma(A)$ is a $\gamma-$regularopen set containing x by (a), such that $G \cap \gamma(A) = \emptyset$. Since $\gamma \in \Gamma_+$, by (c), $G \cap A = \emptyset$, a contradiction to the hypothesis which proves (e).
(f) A is rc-nwdense if and only if $\gamma^\star\gamma(A) = \emptyset$ if and only if $X - \gamma^\star\gamma(A) = X$ if and only if $X - (X - \gamma(X - \gamma(A))) = X$ if and only if $\gamma(X - \gamma(A)) = X$ if and only if $X - \gamma(A)$ is rc-dense. □

Example 2.9 (a) shows that the condition $\gamma \in \Gamma_+$ cannot be dropped to prove the reverse direction in Lemma 2.1 (c). Example 2.1 (b) shows that the subadditivity cannot be dropped in the above Lemma 2.1 (b). Also, it shows that *in an envelope space the intersection of two $\gamma-$regularopen sets need not be a $\gamma-$regularopen set.* Example 2.9 (c) shows that the condition $\gamma \in \Gamma_2$ cannot be dropped in Lemma 2.1 (e). That is, Lemma 2.1 (e) is not true in a weak envelope space.

Example 2.9. (a) Let $X = \{a, b, c\}$ and define $\gamma : \wp(X) \to \wp(X)$ by $\gamma(A) = \{a\}$ for every subset A of X. Then $\mu = \{\{b, c\}\}$ and $\gamma \notin \Gamma_+$. If $A = \{b\}$, then $G \cap \gamma(A) = \emptyset$ for every $G \in \mu$ but $G \cap A \neq \emptyset$.
(b) Consider $X = \{a, b, c\}$ and define $\gamma : \wp(X) \to \wp(X)$ by $\gamma(\emptyset) = \emptyset, \gamma(\{a\}) = \{a\}, \gamma(\{b\}) = \{b\}, \gamma(\{c\}) = \{c\}, \gamma(\{a, b\}) = \{a, b\}, \gamma(\{a, c\}) = \{a, c\}, \gamma(\{b, c\}) = \gamma(X) = X$. Then $\gamma \in \Gamma_+$ and $\gamma \in \Gamma_2$. If $A = \{b\}$ and $B = \{c\}$, then $\gamma(A) \cup \gamma(B) = \{b, c\}$. But $\gamma(A \cup B) = \gamma(\{b, c\}) = X \not\subseteq \{b, c\} = \gamma(A) \cup \gamma(B)$.
Therefore, γ is not subadditive. Here $\xi = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\mu = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. If $U = \{a, b\}$ and $V = \{a, c\}$, then U and V are $\gamma-$regularopen sets but $U \cap V = \{a\}$ is not a $\gamma-$regularopen set.
(c) Consider the monotonic space (X, γ) of Example 2.8. Then $\gamma \in \Gamma_+$, $\gamma \notin \Gamma_2$ and $\mu = \{\emptyset, X, \{b, c\}\}$. If $A = \{b\}$, then $G \cap A \neq \emptyset$ for every nonempty $\gamma-$regularopen set G containing c. Since $\gamma(A) = \gamma(\{b\}) = \{a, b\}$, $c \notin \gamma(A)$.

Theorem 2.6. *Let (X, γ) be an envelope space and $A \subset X$. Then the following hold.*
(a) *If A is wrc-nwdense, then A is rc-nwdense.*
(b) *If γ is subadditive and A is rc-nwdense, then A is wrc-nwdense.*

Proof. (a) Suppose A is not rc-nwdense. Then $G = \gamma^\star\gamma(A) \neq \emptyset$. Since $\gamma \in \Gamma_2$, by Proposition 1.7 (c) of [1], $\gamma^\star \in \Gamma_2$ and so $\gamma^\star(G) = \gamma^\star(\gamma^\star\gamma(A)) = \gamma^\star\gamma(A) = G$ and so G is a nonempty $\gamma-$regularopen set. Since $\gamma \in \Gamma_+$, by Proposition 1.7 (d) of [1], $\gamma^\star \in \Gamma_-$ and so $\gamma^\star\gamma(A) \subset \gamma(A)$ which implies that $G \subset \gamma(A)$. Then for every nonempty $W \in \mu$ with $W \subset G$, $W \cap \gamma(A) \neq \emptyset$ and so by Lemma 2.1 (c), $W \cap A \neq \emptyset$, a contradiction to the hypothesis.
(b) Suppose A is rc-nwdense. Then by Lemma 2.1 (a), $X - \gamma(A)$ is a $\gamma-$regularopen set. By Lemma 2.1 (f), $X - \gamma(A)$ is rc-dense. By Theorem 2.1, $(X - \gamma(A)) \cap V = W$ is nonempty and $W \subset V$ for every nonempty $\gamma-$regularopen set V. By Lemma 2.1 (b), W is $\gamma-$regularopen. Since $\gamma \in \Gamma_+$, $A \cap W = A \cap ((X - \gamma(A)) \cap V) = \emptyset$. Therefore, A is wrc-nwdense. □

Corollary 2.2. *Let (X, γ) be an envelope space, $A \subset X$ and γ be subadditive. Then the following are equivalent.*
(a) *A is wrc-nwdense.*
(b) *A is rc-nwdense.*
(c) *If V is a nonempty $\gamma-$regularopen set, then V is not a subset of $\gamma(A)$.*

Proof. The proof follows from Theorems 2.6, 2.4 and 2.5. □

The following Example 2.10 shows that in Theorem 2.6 (a), the condition *envelope* cannot be replaced by *weak envelope*. Example 2.11 shows that in Theorem 2.6 (b), the condition *subadditive* on γ cannot be dropped.

Example 2.10. Let $X = \{a, b, c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \{a\}$, $\gamma(\{a\}) = \{a, b\}$, $\gamma(\{b\}) = \{a, b\}$, $\gamma(\{c\}) = X$, $\gamma(\{a, b\}) = \gamma(\{a, c\}) = \gamma(\{b, c\}) = \gamma(X) = X$. Then (X, γ) is a weak envelope space. Since $\mu = \{\emptyset\}$, every nonempty subset of X is wrc-nwdense and so $\{c\}$ is wrc-nwdense but is not rc-nwdense.

Example 2.11. Let (X, γ) be an envelope space where $X = \{a, b, c, d\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset$, $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{b\}$, $\gamma(\{c\}) = \{a, b, c\}$, $\gamma(\{d\}) = \{a, d\}$, $\gamma(\{a, b\}) = \gamma(\{a, c\}) = \gamma(\{b, c\}) = \{a, b, c\}$, $\gamma(\{a, d\}) = \{a, d\}$, $\gamma(\{b, d\}) = \gamma(\{c, d\}) = \gamma(X) = X$, $\gamma(\{a, b, c\}) = \{a, b, c\}$, $\gamma(\{a, b, d\}) = \gamma(\{a, c, d\}) = \gamma(\{b, c, d\}) = X$. Then $\mu = \{\emptyset, \{d\}, \{b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. We show that γ is not subadditive. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(A \cup B) = \{a, b, c\}$ and $\gamma(A) \cup \gamma(B) = \{a, b\}$. Therefore, γ is not subadditive. If $A = \{b\}$, then $\gamma^\star\gamma(A) = \gamma^\star(\{b\}) = \emptyset$ and so A is rc-nwdense. If $V = \{b, c\}$, then $V \cap A \neq \emptyset$ and so A is not wrc-nwdense.

A nonempty collection $\mathcal{I}$ of subsets of X is said to be an *ideal* [6] if it satisfies the following. (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$ and (ii) $A \cup B \in \mathcal{I}$ whenever $A \in \mathcal{I}$ and $B \in \mathcal{I}$. In the rest of this section, we discuss some properties of rc-nwdense sets and analyze under what additional conditions on γ, $\mathcal{N}$ is an ideal on X.

Theorem 2.7. *Let (X, γ) be a monotonic space and $A \subset X$. If A is the union of a $\gamma-$regularopen set and an rc-nwdense set, then $A \cap \gamma(X - A)$ is an rc-nwdense set.*

Proof. Let $A = G \cup N$ where G is $\gamma-$regularopen and N is rc-nwdense. If $M = N - G$, then $\gamma^\star\gamma(M) = \gamma^\star\gamma(N - G) \subset \gamma^\star(\gamma(N) \cap \gamma(X - G)) \subset \gamma^\star(\gamma(N)) \cap \gamma^\star(\gamma(X - G)) = \emptyset \cap \gamma^\star(\gamma(X - G)) = \emptyset$ and so M is rc-nwdense. Again, $M \cup G = (N - G) \cup G = N \cup G = A$. Since G is $\gamma-$regularopen such that $G \subset A$, we have $G \subset \gamma^\star(A)$. Now $A \cap \gamma(X - A) = (G \cup N) \cap \gamma((X - G) \cap (X - N)) \subset (G \cup N) \cap \gamma(X - G) = (G \cup N) \cap (X - \gamma^\star(G)) = (G \cup N) \cap (X - G) = N \cap (X - G) = N - G = M$. Since $A \cap \gamma(X - A)$ is a subset of an rc-nwdense set M, $A \cap \gamma(X - A)$ is rc-nwdense. □

The following Theorem 2.8 shows that the converse of Theorem 2.7 is true if the space (X, γ) is an envelope space. Example 2.12 shows that the condition envelope on the space cannot be dropped in Theorem 2.8. Theorem 2.9 gives a characterization of rc-nwdense sets in an envelope space.

Theorem 2.8. *Let (X, γ) be an envelope space and $A \subset X$. If $A \cap \gamma(X - A)$ is an rc-nwdense set, then A is the union of a $\gamma-$regularopen set and an rc-nwdense set.*

Proof. If $A \cap \gamma(X - A) = \emptyset$, then $A \subset X - \gamma(X - A) = \gamma^\star(A)$ which implies that $A = \gamma^\star(A)$, since by Proposition 1.7 (d) of [1], $\gamma \in \Gamma_+$ if and only if $\gamma^\star \in \Gamma_-$. Now $\gamma^\star(A) \cup (A \cap \gamma(X - A)) = A \cup (A \cap (X - \gamma^\star(A)) = A$. Suppose $A \cap \gamma(X - A) \neq \emptyset$. Then $\gamma^\star(A) \cup (A \cap \gamma(X - A)) = (\gamma^\star(A) \cup A)) \cap (\gamma^\star(A) \cup (X - \gamma^\star(A))) = \gamma^\star(A) \cup A$.
Since $\gamma \in \Gamma_2$, $\gamma^\star(A) \in \mu$ and since $\gamma \in \Gamma_+$, $\gamma^\star(A) \subset A$. Therefore, $\gamma^\star(A) \cup (A \cap \gamma(X - A)) = A$. This completes the proof. □

Example 2.12. (a) Let $X = \mathbf{R}$ be the set of all real numbers and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(A) = A$ if $0 \in A$ and $\emptyset$, if otherwise. Then (X, γ) is a monotonic space, $\gamma \in \Gamma_2$, $\mathcal{N} = \{\emptyset\} \cup \{A \mid 0 \notin A\}$ and $\mu = \{\emptyset, \mathbf{R}\} \cup \{A \mid 0 \notin A\}$. If B is a nonempty subset of $\mathbf{R}$ such that $0 \notin B$, then $\gamma(B) = \emptyset$ and so $\gamma \notin \Gamma_+$. If $A = [0, 1)$, then $A \cap \gamma(X - A) = \emptyset$ and so $A \cap \gamma(X - A)$ is rc-nwdense but A is not the union of an rc-nwdense set and a $\gamma-$regularopen set.

(b) Let $X = \{a, b, c\}$ and $\gamma : \wp(X) \to \wp(X)$ be defined by $\gamma(\emptyset) = \emptyset, \gamma(\{a\}) = \{a\}, \gamma(\{b\}) = \{a, b\}, \gamma(\{c\}) = X, \gamma(\{a, b\}) = \gamma(\{a, c\}) = \gamma(\{b, c\}) = \gamma(X) = X$. Then $\gamma \in \Gamma_+$, $\mu = \{\emptyset, \{b, c\}, X\}$ and $\mathcal{N} = \{\emptyset, \{a\}, \{b\}\}$. Since $\gamma(\gamma(\{b\})) \neq \gamma(\{b\}), \gamma \notin \Gamma_2$. If $A = \{a, c\}$, then $A \cap \gamma(X - A) = \{a\}$, which is rc-nwdense but A cannot be written as the union of an rc-nwdense set and a $\gamma-$regularopen set.

Theorem 2.9. *Let (X, γ) be an envelope space and $A \subset X$. Then A is rc-nwdense if and only if $A \subset \gamma(X - \gamma(A))$.*

Proof. If A is *rc-nwdense*, then $\gamma^\star\gamma(A) = \emptyset$. Now, $\gamma(X - \gamma(A)) = X - \gamma^\star\gamma(A) = X \supset A$. Conversely, if $A \subset \gamma(X - \gamma(A))$, then $A \subset X - \gamma^\star\gamma(A)$and so $\gamma^\star\gamma(A) \subset \gamma^\star\gamma(X - \gamma^\star\gamma(A)) = \gamma^\star(X - \gamma^\star\gamma^\star\gamma(A)) = \gamma^\star(X - \gamma^\star\gamma(A)) \subset X - \gamma^\star\gamma(A)$. Therefore, $\gamma^\star\gamma(A) = \emptyset$ which implies that A is rc-nwdense. □

Theorem 2.10. *Let (X, γ) be an envelope space and γ be subadditive. Then the union of two rc-nwdense sets is again an rc-nwdense set and so, if $\mathcal{N}$ is nonempty, then $\mathcal{N}$ is an ideal.*

Proof. Let A and B be two rc-nwdense subsets of X. Then $\gamma^\star\gamma(A) = \emptyset$ and $\gamma^\star\gamma(B) = \emptyset$ and so $X - \gamma^\star\gamma(A) = X$ and $X - \gamma^\star\gamma(B) = X$. This implies that $\gamma(X - \gamma(A)) = X$ and $\gamma(X - \gamma(B)) = X$ and so $X - \gamma(A)$ and $X - \gamma(B)$ are rc-dense sets. Let $G \in \mu$. Since $X - \gamma(A)$ is rc-dense, by Theorem 2.1, $G \cap (X - \gamma(A)) \neq \emptyset$. Since $\gamma \in \Gamma_2$, $X - \gamma(A) \in \mu$ by Lemma 2.1 (a). By Lemma 2.1 (b), $G \cap (X - \gamma(A)) \in \mu$. Since $X - \gamma(B)$ is rc-dense, $(G \cap (X - \gamma(A))) \cap (X - \gamma(B)) \neq \emptyset$ which implies that $G \cap (X - (\gamma(A) \cup \gamma(B))) \neq \emptyset$ and so $G \cap (X - (\gamma(A \cup B))) \neq \emptyset$, since γ is additive. By Corollary 2.1, $X - (\gamma(A \cup B))$ is rc-dense and so $\gamma(X - (\gamma(A \cup B))) = X$ which implies that $X - \gamma(X - (\gamma(A \cup B))) = \emptyset$. Therefore, $\gamma^\star\gamma(A \cup B) = \emptyset$ and so $A \cup B$ is rc-nwdense.
If $\mathcal{N}$ is nonempty, since $\gamma^\star\gamma \in \Gamma$, every subset of an rc-nwdense set is again an rc-nwdense set. Hence it follows that $\mathcal{N}$ is an ideal. □

Example 2.12 (a) above shows that if the finite union of rc-nwdense subsets of the space (X, γ) is rc-nwdense, then γ need not be additive. The following Example 2.13 shows that the condition subadditive on γ in Theorem 2.10 cannot be dropped.

Example 2.13. Consider the monotonic space (X, γ) of Example 2.10. Then $\gamma \in \Gamma_+$ and $\gamma \notin \Gamma_2$. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(A) \cup \gamma(B) = \{a, b\}$. But $\gamma(A \cup B) = \gamma(\{a, b\}) = X \nsubseteq \{a, b\} = \gamma(A) \cup \gamma(B)$. Therefore, γ is not subadditive. Here $\xi = \{X\}$ and $\mu = \{\emptyset\}$. Also, $\mathcal{N} = \{\emptyset, \{a\}, \{b\}\}$. If $C = \{a\}$ and $D = \{b\}$, then $C \cup D = \{a, b\}$ and $\gamma^\star\gamma(C \cup D) = \gamma^\star(X) = X - \gamma(\emptyset) = X - \{a\} = \{b, c\} \neq \emptyset$ and so $C \cup D$ is not an rc-nwdense set.

Theorem 2.11. *Let (X, γ) be an envelope space, $A \subset X$ and γ be subadditive. Then the following hold.*
(a) *$\gamma(A) \cap V \subset \gamma(A \cap V)$ for every $\gamma-$regularopen set V.*
(b) *$\gamma(\gamma(A) \cap V) = \gamma(A \cap V)$ for every $\gamma-$regularopen set V.*
(c) *If A is rc-dense, then $\gamma(V) = \gamma(A \cap V)$ for every $\gamma-$regularopen set V.*

Proof. (a) Suppose $x \in \gamma(A) \cap V$. Then $x \in \gamma(A)$ and $x \in V$. If G is a $\gamma-$regularopen set containing x, by Lemma 2.1 (b), $G \cap V$ is a $\gamma-$regularopen set containing x and so by Lemma 2.1 (d), $(G \cap V) \cap A = G \cap (V \cap A) \neq \emptyset$. By Lemma 2.1 (e), $x \in \gamma(V \cap A)$. Therefore, $\gamma(A) \cap V \subset \gamma(A \cap V)$.

(b) Since $\gamma \in \Gamma_+$, $A \cap V \subset \gamma(A) \cap V$ and so $\gamma(A \cap V) \subset \gamma(\gamma(A) \cap V) \subset \gamma\gamma(A \cap V) = \gamma(A \cap V)$ and so $\gamma(\gamma(A) \cap V) = \gamma(A \cap V)$.

(c) By (b), if V is $\gamma-$regularopen, then $\gamma(A \cap V) = \gamma(\gamma(A) \cap V) = \gamma(X \cap V) = \gamma(V)$ and so (c) follows. □

Let $\lambda \subset \wp(X)$. $\gamma \in \Gamma$ is said to be $\lambda-$*friendly* [5] if $\gamma(A) \cap V \subset \gamma(A \cap V)$ for every subset A of X and $V \in \lambda$. A generalized topology ψ is said to be a *quasi-topology* [5] if $M \cap M_1 \in \psi$ whenever $M \in \psi$ and $M_1 \in \psi$.

Theorem 2.12. *Let (X, γ) be a monotonic space. Then the following hold.*
(a) *If γ is a weak envelope, then μ is a generalized topology. In addition, if γ is subadditive, then μ is a quasi-topology.*
(b) *If γ is an envelope, then γ is subadditive if and only if γ is $\mu-$friendly.*
(c) *γ is subadditive if and only if $\gamma^\star(A \cap B) = \gamma^\star(A) \cap \gamma^\star(B)$ for every subsets A and B of X.*
(d) *If γ is an envelope, γ is subadditive, G is $\gamma-$regularopen and $A \subset X$, then $G \cap \gamma^\star(A) = \gamma^\star(G \cap A)$.*
(e) *If γ is an envelope, γ is subadditive, F is $\gamma-$regularclosed and $A \subset X$, then $\gamma^\star(A \cup F) \subset \gamma^\star(A) \cup F$.*
(f) *If γ is an envelope, γ is subadditive, F is $\gamma-$regularclosed and $A \subset X$, then $\gamma(A \cup F) = \gamma(A) \cup F$.*

Proof. (a) By Lemmas 1.3 and 1.4 of [3], μ is a generalized topology. By Lemma 2.1 (b), μ is a quasi-topology.

(b) If γ is subadditive, then by Theorem 2.11 (a), γ is $\mu-$friendly. Conversely, suppose γ is $\mu-$friendly. For subsets A and B of X, $\gamma(A \cup B) - \gamma(B) = \gamma(A \cup B) \cap (X - \gamma(B))$. Since $X - \gamma(B) \in \mu$ by Lemma 2.1 (a), by Theorem 2.11 (a), $\gamma(A \cup B) - \gamma(B) \subset \gamma((A \cup B) \cap (X - \gamma(B))) = \gamma(A \cap (X - \gamma(B))) \subset \gamma(A)$ and so $\gamma(A \cup B) \subset \gamma(A) \cup \gamma(B)$.

(c) The proof follows from the definition of $\gamma^\star$.

(d) Let G be $\gamma-$regularopen and A be any subset of X. Then $G \cap \gamma^\star(A)$ is a $\gamma-$regular-open set by Lemma 2.1 (b), such that $G \cap \gamma^\star(A) \subset G \cap A$, since $\gamma^\star \in \Gamma_-$. Therefore, $G \cap \gamma^\star(A) \subset \gamma^\star(G \cap A) = \gamma^\star(G) \cap \gamma^\star(A) = G \cap \gamma^\star(A)$, by (c). Therefore, $G \cap \gamma^\star(A) = \gamma^\star(G \cap A)$.

(e) Now $X - \gamma^\star(A \cup F) = \gamma(X - (A \cup F)) = \gamma((X - A) \cap (X - F)) \supset \gamma(X - A) \cap (X - F)$, by Theorem 2.11 (a). Therefore, $X - \gamma^\star(A \cup F) \supset (X - \gamma^\star(A)) \cap (X - F) = X - (\gamma^\star(A) \cup F)$ and so $\gamma^\star(A \cup F) \subset \gamma^\star(A) \cup F$.

(f) Now $X - \gamma(A \cup F) = \gamma^\star(X - (A \cup F)) = \gamma^\star((X - A) \cap (X - F)) = \gamma^\star(X - A) \cap (X - F) = (X - \gamma(A)) \cap (X - F) = X - (\gamma(A) \cup F)$ and so $\gamma(A \cup F) = \gamma(A) \cup F$. □

The following Example 2.14 shows that in a weak envelope space (X, γ), if γ is $\mu-$friendly, then γ need not be subadditive. Also, this example shows that the converse of Lemma 2.1 (b) is not true.

Example 2.14. Let (X, γ) be the space of Example 2.10. Then $\mu = \{\emptyset\}$. Since $\gamma(A) \cap V \subset \gamma(A \cap V)$ for every $\gamma-$regularopen set V and $A \subset X$, γ is $\mu-$friendly. If $A = \{a\}$ and $B = \{b\}$, then $\gamma(\{a\}) = \{a, b\}$ and $\gamma(\{b\}) = \{a, b\}$ so that $\gamma(A) \cup \gamma(B) = \{a, b\}$. But $\gamma(A \cup B) = X$ and so γ is not subadditive.

Acknowledgement. The author sincerely thank Prof. D. Sivaraj, D. J. Academy for Managerial Excellence, Coimbatore - 641 032, India for his suggestions and guidance and the referee for the valuable suggestions and comments.

References

[1] Császár, Á., *Generalized Open Sets*, Acta Math. Hungar. **75** (1997), No. 1-2, 65-87
[2] Császár, Á., *Generalized topology, generalized continuity*, Acta Math. Hungar. **96** (2002), 351-357
[3] Császár, Á., *Separation axioms for generalized topologies*, Acta Math. Hungar. **104** (2004), 63-69
[4] Császár, Á., *Ultratopologies generated by generalized topologies*, Acta Math. Hungar. **110** (2006), 153-157
[5] Császár, Á., *Further remarks on the formula for $\gamma-$interior*, Acta Math. Hungar. **113** (2006), 325-332
[6] Kuratowski, K., *Topology*, **I**, New York, Academic Press, 1966

Department of Mathematics
ANJA College
Sivakasi - 626 124, Tamil Nadu, India
E-mail address: renu_siva2003@yahoo.com