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Weak stability of iterative procedures for some coincidence theorems

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ABSTRACT.
The purpose of this paper is to study the problem of weak stability of common fixed point iterative procedures for some classes of contractive type mappings. An example to illustrate weakly stable but not stable iterative fixed point procedures is also given.

1. INTRODUCTION

In [25], Singh and Prasad have studied the (S,T)-stability of iterative procedures

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \dots , \tag{1.1}$$

for some classes of contractive type mappings.

Namely, in the metric space X, for $A, B, S, T : Y \to X$, $\forall x, y \in Y$, where $q \in (0, 1)$, they are using the following conditions :

$$d(Tx, Ty) \le qd(Sx, Sy); \tag{1.2}$$

$$d(Tx,Ty) \le q \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{[d(Sx,Ty)+d(Sy,Tx)]}{2}\right\}; \tag{1.3}$$

$$d(Tx,Ay) \le q \max\left\{d(Sx,By), d(Sx,Tx), d(By,Ay), \frac{[d(Sx,Ay)+d(By,Tx)]}{2}\right\}; \tag{1.4}$$

$$d(Tx,Ay) \le q \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ay), \frac{[d(Sx,Ay)+d(Sy,Tx)]}{2}\right\}. \tag{1.5}$$

For $Y = X$ and $f(T, x_n) = Tx_n$, the iterative procedure (1.1) yields the Jungck iterations, namely $Sx_{n+1} = Tx_n$, $n = 0, 1, \dots$. This procedure was essentially introduced by Jungck and it becomes the Picard iterative procedure when $S = id$, the identity map on X.

Jungck showed that the maps S and T satisfying (1.2), for all $x, y \in X$ have an unique common fixed point in complete X, provided that S and T are commuting, $T(X) \subseteq S(X)$, and S is continuous. However, the following significantly improved version of this result is generally called the Jungck contraction principle.

Theorem 1.1. [24] *Let $S, T : Y \to X$ satisfy (1.2) $\forall x, y \in Y$. If $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of X, then S and T have a coincidence and for any x_0 in Y, there exists a sequence $\{x_n\}$ in Y such that*

(1) $Sx_{n+1} = Tx_n$, $n = 0, 1, \dots$,

(2) $\{Sx_n\}$ converges to Sz for some z in Y, and $Sz = Tz$, that is, S and T have a coincidence at z.

Further, if $Y = X$ and S, T commute (just) at z, then S and T have an unique fixed point.

The concept of stability they used is given by the next definition:

Definition 1.1. [25] Let (X, d) be a metric space and $Y \subseteq X$.

Let $S, T : Y \to X$, $T(Y) \subseteq S(Y)$ and z a coincidence point of T and S, that is a point for which we have $Sz = Tz = u \in X$. For any $x_0 \in Y$, let the sequence $\{Sx_n\}$ generated by the general iterative procedure

$$Sx_{n+1} = f(T, x_n), \quad n = 1, 2, \dots, \tag{1.6}$$

converge to an element. Let $\{Sy_n\} \subset X$ be an arbitrary sequence, and set

$$\varepsilon_n = d(Sy_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, \dots .$$

Then the iterative procedure $f(T, x_n)$ is (S,T)-stable or stable with respect to (S,T) if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ implies that } \lim_{n\to\infty} Sy_n = u.$$

Their main stability result is the next theorem:

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Theorem 1.2. [25] *Let (X, d) be a metric space and S, T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, that is, $Tz = Sz = u$.*

For any $x_0 \in Y$, let the sequence $\{Sx_n\}$ generated by the iterative procedure $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to u. Let $\{Sy_n\} \subset X$ and then define $\varepsilon_n = d(Sy_{n+1}, Ty_n)$, $n = 0, 1, 2, ...$ If the pair (S, T) is a J-contraction with q as J-constant, that is, S and T satisfy (1.2) for all $x, y \in Y$ and $q < 1$, then

$$d(u, Sy_{n+1}) \le d(u, Sx_{n+1}) + q^{n+1} d(Sx_0, Sy_0) + \sum_{r=0}^{n} q^{n-r} \varepsilon_r.$$

Further,

$$\lim_{n\to\infty} Sy_n = u \quad \text{if and only if} \quad \lim_{n\to\infty} \varepsilon_n = 0.$$

In other words, Theorem 1.2 shows that the iterative procedure $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ is stable with respect to (S, T).

The definition of (S, T)-stability iterative procedures used in [25], like that in [5] and [6] is based on the choice of an *arbitrary* sequence $\{Sy_n\}$. However, as shown in [?], chapter 7, by considering an arbitrary sequence in Definition 1.1 we do not treat the problem of stability in its general context.

The aim of this note is to restate all results in the paper of [25] in the case of the concept of weak stability.

2. Preliminaries

For our purposes here, we need some coincidence theorems for maps on an arbitrary nonempty set Y with values in a metric space X.

Theorem 2.3. [25] *Let (X, d) be a metric space and $S,\ T : Y \to X$ such that $T(Y) \subseteq S(Y)$ and (1.2) holds with $q < 1$. If $S(Y)$ or $T(Y)$ is a complete subspace of X, then S and T have a coincidence.*

Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that $S_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ and $\{Sx_n\}$ converges to Sz for some z in Y and $Sz = Tz$, that is, S and T have a coincidence at z.

Further, if $Y = X$ and S and T commute (just) at z, then S and T have an unique common fixed point.

Theorem 2.4. [25] *Let (X, d) be a metric space and $A, B, S, T : Y \to X$ such that $T(Y) \subseteq B(Y)$, $A(Y) \subseteq S(Y)$ and (1.4) holds, with $q < 1$ and $\lambda < 1$, where $\lambda = max\left\{q, \dfrac{q}{2-q}\right\}$. If one of $A(Y)$, $B(Y)$, $S(Y)$ or $T(Y)$ is a complete subspace of X, then*

1. *T and S have a coincidence, i.e. there exists a $v \in Y$ such that $Sv = Tv$;*
2. *A and B have a coincidence, i.e. there exists a $w \in Y$ such that $Bw = Aw$.*
   Further, if $Y = X$, then
3. *T and S have a common fixed point provided that T and S commute (just) at the coincidence point v;*
4. *A and B have a common fixed point provided that A and B commute (just) at the coincidence point w;*
5. *S, T, A and B have a common fixed point provided (3) and (4) both are true.*

Corollary 2.1. *Let (X, d) be a metric space and $A,\ B,\ S,\ T\ : Y \to X$ such that $A(Y) \cup T(Y) \subseteq S(Y)$ and the following holds*

$$d(Tx, Ay) \le q \max\left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ay), \frac{[d(Sx, Ay) + d(Sy, Tx)]}{2}\right\}$$

with $q < 1$ and $\lambda < 1$, where $\lambda = max\left\{q, \dfrac{q}{2-q}\right\}$.

If one of $A(Y)$, $S(Y)$ or $T(Y)$ is a complete subspace of X, then there exists a z such that $Az = Tz = Sz$.

Further, if $Y = X$, then

1. *T and S have a common fixed point, provided that T and S commute (just) at z;*
2. *A and S have a common fixed point, provided that A and S commute (just) at z;*
3. *A, T and S have a common fixed point, provided that S commute with each of A and T (just) at z.*

In the Theorem 2.4, for $T = A$ and $S = B$, we have the following:

Corollary 2.2. *Let (X, d) be a metric space and $S,\ T\ : Y \to X$ such that $T(Y) \subseteq S(Y)$ and (1.3) holds with $q < 1$. If $S(Y)$ or $T(Y)$ is a complete subspace of X, then S and T have a coincidence.*

Indeed, for any $x_0 \in Y$, there exists a sequence $\{x_n\}$ in Y such that $S_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ and $\{Sx_n\}$ converges to Sz for some z in Y and $Sz = Tz$, that is, S and T have a coincidence at z.

Further, if $Y = X$ and S and T commute (just) at z, then S and T have an unique common fixed point.

Lemma 2.1. [?] *Let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers. Then,*

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ iif } \lim_{n\to\infty} s_n = 0, \text{ where } s_n = \sum_{i=0}^{n} k^{n-i} \varepsilon_i \text{ and } k \in [0, 1).$$

3. Weak stability of iterative procedures

The concept of stability is not very precise because of the sequence $\{y_n\}_{n=0}^{\infty}$ which is *arbitrary* taken. It would be more natural that $\{y_n\}$ to be an *approximate sequence* of $\{x_n\}$ and then to introduce a weaker concept of stability, called *weak stability*. Therefore, any stable iteration will be also weakly stable but he reverse is not generally true.

Definition 3.2. [?] Let (X,d) be a metric space and $\{x_n\}_{n=1}^{\infty} \subset X$ be a given sequence. We shall say that $\{y_n\}_{n=0}^{\infty} \in X$ is an approximate sequence of $\{x_n\}$ if, for any $k \in \mathbb{N}$, there exists $\eta = \eta(k)$ such that

$$d(x_n, y_n) \leq \eta, \text{ for all } n \geq k.$$

Remark 3.1. We can have approximate sequences of both convergent and divergent sequences.

The following result will be useful in the sequel.

Lemma 3.2. [?] *The sequence $\{y_n\}$ is an approximate sequence of $\{x_n\}$ if and only if there exists a decreasing sequence of positive numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ such that*

$$d(x_n, y_n) \leq \eta_n, \textit{ for any } n \geq k \textit{ (fixed).}$$

Definition 3.3. [?] Let (X,d) be a metric space and $T : X \to X$ be a map. Let $\{x_n\}$ be an iteration procedure defined by $x_0 \in X$ and

$$x_{n+1} = f(T, x_n), \quad n \geq 0. \tag{3.7}$$

Suppose that $\{x_n\}$ converges to a fixed point p of T. If for any approximate sequence $\{y_n\} \subset X$ of $\{x_n\}$

$$\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$$

implies

$$\lim_{n\to\infty} y_n = p,$$

then we shall say that (3.7) is weakly T-stable or weakly stable in respect to T.

Further on, we will present some examples from [?] in order to show how important and natural is to restrict the stability concept to approximate sequences $\{y_n\}$ of $\{x_n\}$.

Example 3.1. [?] Let $T : \mathbb{R} \to \mathbb{R}$ be given by $Tx = \frac{1}{2}x$, where $\mathbb{R}$ is endowed with the usual metric. As T is an $\frac{1}{2}$-contraction, the Ishikawa iteration $\{x_n\}_{n=1}^{\infty}$ is T-stable, hence almost T-stable and weakly T-stable, too.

To prove the fact that the Ishikawa iteration is not T-stable, in [?] is used the sequence $\{y_n\}_{n=1}^{\infty}$ given by $y_n = \frac{n}{1+n}$, $n \geq 0$.

But this is obviously nonsense, because $x_n \to 0$, the unique fixed point of T, while $y_n \to 1$ as $n \to \infty$, although, by construction, $\{y_n\}_{n=1}^{\infty}$ would have to be an approximate sequence of $\{x_n\}$.

Example 3.2. [?] Let $T : [0,1] \to [0,1]$ be given by

$$Tx = \begin{cases} \frac{1}{2}, & x \in \left[0, \frac{1}{2}\right] \\ 0, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

where $\left[0, 1\right]$ is endowed with the usual metric. We have $F_T = \left\{\frac{1}{2}\right\}$ and we know that the Picard iteration is not T-stable.

In [?] is showed that the Picard iteration is also not weakly T-stable.

Let $x_0 \in [0,1]$ and $x_{n+1} = Tx_n$, for $n = 1, 2, \ldots$. If $x_0 \in \left[0, \frac{1}{2}\right]$, then $x_1 = Tx_0 = \frac{1}{2}$ and if $x_0 \in \left(\frac{1}{2}, 1\right]$, then $x_1 = Tx_0 = 0$. Results that $x_n = \frac{1}{2}$ for all $n \geq 2$, hence

$$\lim_{n\to\infty} x_n = \frac{1}{2} = T\left(\frac{1}{2}\right).$$

Let $\{y_n\}$ be an approximate sequence of $\{x_n\}$. There exists a decreasing sequence of positive numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ such that

$$|x_n - y_n| \leq \eta_n, \quad \text{for } n \geq k.$$

In particular, we can take $y_n = x_n + (-1)^n \cdot \eta_n$, $n \geq k$ which is equivalent to

$$y_n = \frac{1}{2} + (-1)^n \eta_n, \quad \text{for each } n \geq 2.$$

Then

$$Ty_n = \begin{cases} \frac{1}{2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

and hence $|y_{n+1}-Ty_n| = |y_{n+1}-\frac{1}{2}|$ if $n=2p+1$ is odd and $|y_{n+1}-Ty_n|=0$ if $n=2p$ is even. By $\lim_{n\to\infty}|y_{n+1}-Ty_n|=0$ results

$$\lim_{p\to\infty} y_{2p+2} = \frac{1}{2} \text{ and } \lim_{p\to\infty} y_{2p+1} = 0.$$

This shows that $\{y_n\}$ is not convergent and this implies that the Picard iteration is not weakly T-stable.

Example 3.3. [?] Let $T:[0,1]\to[0,1]$ be given by

$$Tx = \begin{cases} 0, & x\in\left[0,\frac{1}{2}\right] \\ \frac{1}{2}, & x\in\left(\frac{1}{2},1\right] \end{cases}$$

where $[0,1]$ is endowed with the usual metric.

Let $x_0\in[0,1]$ and $x_{n+1}=Tx_n$, for $n=1,2,\ldots$. If $x_0\in\left[0,\frac{1}{2}\right]$, then $x_1=Tx_0=0$ and if $x_0\in\left(\frac{1}{2},1\right]$, then $x_1=Tx_0=\frac{1}{2}$. Results that $x_n=0$ for all $n\geq 2$, hence $\lim_{n\to\infty}x_n=0=T(0)$.

Let $\{y_n\}$ be an approximate sequence of $\{x_n\}$. There exists a decreasing sequence of positive numbers $\{\eta_n\}$ converging to some $\eta\geq 0$ such that

$$|x_n-y_n|\leq\eta_n, \text{ for } n\geq 0.$$

Then

$$x_n-\eta_n\leq y_n\leq x_n+\eta_n, \ \ n\geq 0.$$

Since $x_n=0$ for all $n\geq 2$, we obtain $0\leq y_n\leq\eta_n$ for all $n\geq 2$. So, we can choose $\{\eta_n\}$ such that $\eta_n\leq\frac{1}{2}$ for all $n\geq 2$. Then $Ty_n=0$, $n\geq 2$ and by $\lim_{n\to\infty}|y_{n+1}-Ty_n|=0$ results

$$\lim_{n\to\infty}y_n=0=T(0).$$

This shows that the Picard iteration is weakly T-stable.

Now, let us show that the Picard iteration is not T-stable. Indeed, if we take $\{y_n\}$, $y_n=\frac{n+2}{2n}$, $n\geq 1$, then

$$\varepsilon_n=|y_{n+1}-Ty_n|=\left|\frac{n+3}{2(n+1)}-\frac{1}{2}\right|,$$

because $y_n\geq\frac{1}{2}$, for $n\geq 1$.

Therefore, $\lim_{n\to\infty}\varepsilon_n=0$ but $\lim_{n\to\infty}y_n=\frac{1}{2}$, so the Picard iteration is not T-stable.

The concept of T-stability now will be transposed to (S,T)-stability in a metric space.

Definition 3.4. Let (X,d) be a metric space and $Y\subseteq X$.

Let $S,\ T:Y\to X$ be such as $T(Y)\subseteq S(Y)$ and z is a coincidence point of S and T, that is, $Sz=Tz=u$.

For any $x_0\in Y$, let the sequence $\{Sx_n\}$ be generated by the general iterative procedure

$$Sx_{n+1}=f(T,x_n), \ \ n=1,2,..., \tag{3.8}$$

convergent to an element $u\in X$.

If for any approximate sequence $\{Sy_n\}\subset X$ of $\{Sx_n\}$, we have that

$$\lim_{n\to\infty}d(Sy_{n+1},f(T,y_n))=0$$

implies

$$\lim_{n\to\infty}Sy_n=u,$$

then we shall say that (3.8) is weakly (S,T)-stable or weakly stable with respect to (S,T).

4. Weak stability results

A basic result for the stability of J-iterations, i.e., Theorem 1.2 will be transposed to weak stability.

Theorem 4.5. *Let (X,d) be a metric space and S,T maps on an arbitrary set Y with values in X such that $T(Y)\subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, that is, $Tz=Sz=u$.*

For any $x_0\in Y$, let the sequence $\{Sx_n\}$ generated by the iterative procedure $Sx_{n+1}=Tx_n$, $n=0,1,2,\ldots$, converge to u. Let $\{Sy_n\}\subset X$ be an approximate sequence of $\{Sx_n\}$ and then define $\varepsilon_n=d(Sy_{n+1},Ty_n),\ n=0,1,2,\ldots$ If the pair (S,T) is a J-contraction with q as J-constant, that is, S and T satisfy (1.2) for all $x,y\in Y$ and $q<1$, then

(1) $d(u,Sy_{n+1})\le d(u,Sx_{n+1})+q^{n+1}d(Sx_0,Sy_0)+\sum\limits_{r=0}^{n}q^{n-r}\varepsilon_r$;

(2) $\lim\limits_{n\to\infty}Sy_n=u$ *if and only if* $\lim\limits_{n\to\infty}\varepsilon_n=0$.

Proof. By the triangle inequality and the condition (1.2), we have

$$d(u,Sy_{n+1})\le d(u,Sx_{n+1})+qd(Sx_n,Sy_n)+\varepsilon_n\ \le$$
$$\le d(u,Sx_{n+1})+q\left[d(Tx_{n-1},Ty_{n-1})+d(Ty_{n-1},Sy_n)\right]+\varepsilon_n.$$

After $n-1$ steps of this process, yields (1).

To prove (2), first suppose that $\lim_{n\to\infty}Sy_n=u$. Then,

$$\varepsilon_n=d(Sy_{n+1},Ty_n)\le d(Sy_{n+1},Sx_{n+1})+d(Tx_n,Ty_n)\le$$
$$\le d(Sy_{n+1},Sx_{n+1})+qd(Sx_n,Sy_n)\to 0,\ as\ n\to\infty.$$

Now, suppose that $\lim\limits_{n\to\infty}\varepsilon_n=0$. Then, $\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n}q^{n-i}\varepsilon_i=0$. Also, $\lim\limits_{n\to\infty}Sx_n=u$ implies that $\lim\limits_{n\to\infty}d(u,Sx_{n+1})=0$. □

Theorem 4.6. *Let (X,d) be a metric space and S,T maps on an arbitrary set Y with values in X such that $T(Y)\subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, that is, $Tz=Sz=u$.*

For any $x_0\in Y$, let the sequence $\{Sx_n\}$ generated by the iterative procedure $Sx_{n+1}=Tx_n$, $n=0,1,2,\ldots$, converge to u. Let $\{Sy_n\}\subset X$ be an approximate sequence of $\{Sx_n\}$ and then define $\varepsilon_n=d(Sy_{n+1},Ty_n),\ n=0,1,2,\ldots$ If the pair (S,T) satisfy

$$d(Tx,Ty)\le qd(Sx,Sy)+Ld(Sx,Tx) \tag{4.9}$$

for all $x,y\in Y$, where $q\in(0,1)$ and $L\ge 0$, then

(1) $d(u,Sy_{n+1})\le d(u,Sx_{n+1})+q^{n+1}d(Sx_0,Sy_0)+L\sum_{r=0}^{n}q^{n-r}d(Sx_r,Tx_r)+\sum\limits_{r=0}^{n}q^{n-r}\varepsilon_r$;

(2) $\lim\limits_{n\to\infty}Sy_n=u$ *if and only if* $\lim\limits_{n\to\infty}\varepsilon_n=0$.

Proof. From (4.9), for any nonnegative integer n, we have

$$d(Sx_{n+1},Sy_{n+1})=d(Tx,Sy_{n+1})\le d(Tx_n,Ty_n)+d(Ty_n,Sy_{n+1})\le$$
$$\le qd(Sx_n,Sy_n)+Ld(Sx_n,Tx_n)+\varepsilon_n\le$$
$$\le q^2d(Sx_{n-1},Sy_{n-1})+qLd(Sx_{n-1},Tx_{n-1})+Ld(Sx_n,Tx_n)+q\varepsilon_{n-1}+\varepsilon_n.$$

After $n-1$ steps, we obtain

$$d(Sx_{n+1},Sy_{n+1})\le q^{n+1}d(Sx_0,Sy_0)+L\sum_{r=0}^{n}q^{n-r}d(Sx_r,Ty_r)+\sum_{r=0}^{n}q^{n-r}\varepsilon_r.$$

Therefore,

$$d(u,Sy_{n+1})\le d(u,Sx_{n+1})+d(Sx_{n+1},Sy_{n+1})\le$$
$$\le d(u,Sx_{n+1})+q^{n+1}d(Sx_0,Sy_0)+L\sum_{r=0}^{n}q^{n-r}d(Sx_r,Ty_r)+\sum_{r=0}^{n}q^{n-r}\varepsilon_r.$$

This provides (1).

Now, assume that $\lim\limits_{n\to\infty}Sy_n=u$. Then,

$$\varepsilon_n=d(Sy_{n+1},Ty_n)\le d(Sy_{n+1},Sx_{n+1})+d(Tx_n,Ty_n)\le$$
$$\le d(Sy_{n+1},Sx_{n+1})+qd(Sx_n,Sy_n)+Ld(Sx_n,Tx_n).$$

For $n\to\infty$, we obtain that $\varepsilon_n\to 0$.

Now, suppose that $\lim\limits_{n\to\infty}\varepsilon_n=0$. Since $q\in(0,1)$ and $\lim\limits_{n\to\infty}Sx_n=u$, applying Lemma 2.1 and passing (1) to the limit, we obtain

$$\lim_{n\to\infty}d(u,Sy_{n+1})\le\lim_{n\to\infty}\left\{L\sum_{r=0}^{n}q^{n-r}d(Sx_r,Tx_r)+\sum_{r=0}^{n}q^{n-r}\varepsilon_r\right\}. \tag{4.10}$$

Let A denote the lower triangular matrix with entries $a_{nr} = q^{n-r}$. Then, $\lim\limits_{n\to\infty} a_{nr} = 0$ for each r and $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n} a_{nr} = \frac{1}{1-q}$.

For any convergent sequence $\{s_n\}$, we have $\lim\limits_{n\to\infty} A(s_n) = \dfrac{1}{1-q}$. Thus, the right side of (4.10) vanishes. □

Theorem 4.7. *Let (X,d) be a metric space and S, T maps on an arbitrary set Y with values in X such that $T(Y) \subseteq S(Y)$, and $S(Y)$ or $T(Y)$ is a complete subspace of X. Let z be a coincidence point of T and S, that is, $Tz = Sz = u$.*

For any $x_0 \in Y$, let the sequence $\{Sx_n\}$ generated by the iterative procedure $Sx_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, converge to u. Let $\{Sy_n\} \subset X$ be an approximate sequence of $\{Sx_n\}$ and then define $\varepsilon_n = d(Sy_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots$ If the pair (S,T) satisfy (1.3) for all $x, y \in Y$ and $q \in (0,1)$, then

(1) $d(u, Sy_{n+1}) \leq d(u, Sx_{n+1}) + \alpha^n d(Sx_1, Sy_1) + \alpha \sum\limits_{r=0}^{n} \alpha^{n-r} d(Sx_r, Tx_r) + + \sum\limits_{r=0}^{n} \alpha^{n-r} \varepsilon_r$, *where* $\alpha = \dfrac{q}{1-q}$;

(2) $\lim\limits_{n\to\infty} Sy_n = u$, *if and only if* $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Proof. From (1.3), for any $x, y \in Y$, one of the following is true:

- $d(Tx, Ty) \leq q d(Sx, Sy)$,
- $d(Tx, Ty) \leq q d(Sx, Tx)$,
- $d(Tx, Ty) \leq q d(Sx, Ty) \leq q\left[d(Sy, Sx) + d(Sx, Ty)\right] \leq$
$$\leq \frac{k}{1-k}\left[d(Sx, Sy) + d(Sx, Tx)\right], \text{ and}$$
- $d(Tx, Ty) \leq \dfrac{q}{2}\left[d(Sx, Ty) + d(Sy, Tx)\right] \leq$
$$\leq \frac{k}{2-k} d(Sx, Sy) + \frac{2k}{2-k} d(Sx, Tx).$$

Therefore, in all cases, we get
$$d(Tx, Ty) \leq \alpha\left[d(Sx, Sy) + d(Sx, Tx)\right].$$

For any nonnegative integer n, we get
$$d(Sx_{n+1}, Sy_{n+1}) = d(Sx_n, Sy_{n+1}) \leq d(Tx_n, Ty_n) + d(Ty_n, Sy_{n+1}) \leq$$
$$\leq \alpha\left[d(Sx_n, Sy_n) + d(Sx_n, Tx_n)\right] + \varepsilon_n \leq$$
$$\leq \alpha^2 d(Sx_{n-1}, Sy_{n-1}) + \alpha^2 d(Sx_{n-1}, Tx_{n-1}) + \alpha d(Sx_n, Tx_n) + \alpha\varepsilon_{n-1} + \varepsilon_n.$$

After $n-1$ steps, we obtain
$$d(Sx_{n+1}, Sy_{n+1}) \leq \alpha^{n+1} d(Sx_0, Sy_0) + \alpha \sum_{r=0}^{n} \alpha^{n-r} d(Sx_r, Ty_r) + \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r.$$

Therefore
$$d(u, Sy_{n+1}) \leq d(u, Sx_{n+1}) + d(Sx_{n+1}, Sy_{n+1}) \leq$$
$$\leq d(u, Sx_{n+1}) + \alpha^{n+1} d(Sx_0, Sy_0) + \alpha \sum_{r=0}^{n} \alpha^{n-r} d(Sx_r, Tx_r) + \sum_{r=0}^{n} \alpha^{n-r} \varepsilon_r.$$

This proves (1).

Now, assume that $\lim\limits_{n\to\infty} Sy_n = u$. Then,
$$\varepsilon_n = d(Sy_{n+1}, Ty_n) \leq d(Sx_{n+1}, Sy_{n+1}) + d(Tx_n, Ty_n) \leq$$
$$\leq d(Sx_{n+1}, Sy_{n+1}) + \alpha\left[d(Sx_n, Sy_n) + d(Sx_n, Tx_n)\right].$$

Letting $n \to \infty$, we obtain $\varepsilon_n \to 0$.

Suppose that $\lim\limits_{n\to\infty} \varepsilon_n = 0$. Since $\alpha \in (0,1)$ and $\lim\limits_{n\to\infty} Sx_n = u$, applying L. 2.1 and passing (1) to the limit, we obtain
$$\lim_{n\to\infty} d(u, Sy_{n+1}) \leq \lim_{n\to\infty} \alpha \sum_{r=0}^{n} \alpha^{n-r} d(Sx_r, Tx_r). \tag{4.11}$$

Let A denote the lower triangular matrix with entries $a_{nr} = \alpha^{n-r}$. Then, $\lim\limits_{n\to\infty} a_{nr} = 0$ for each r and $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n} a_{nr} = \frac{1}{1-\alpha}$.

For any convergent sequence $\{s_n\}$, we have $\lim\limits_{n\to\infty} A(s_n) = \dfrac{1}{1-\alpha}$. Thus, the right side of (4.11) vanishes. □

5. Example

Let $S,T : [0,1] \to [0,1]$ be given by $Tx = 0$, if $x \in \left[0, \frac{1}{2}\right]$ and $Tx = \frac{1}{2}$, if $x \in \left(\frac{1}{2}, 1\right]$ and $Sx = x$, where $[0,1]$ is endowed with the usual metric.

We will show that the Picard iteration is not (S,T)-stable but it is (S,T)-weakly stable.

In order to prove the first claim, let (Sy_n) be given by $Sy_n \equiv y_n = \frac{n+2}{2n}$, $n \geq 1$.

Then

$$\varepsilon_n = |Sy_{n+1} - f(T,x_n)| = |y_{n+1} - Ty_n| = |\frac{n+3}{2(n+1)} - \frac{1}{2}|,$$

because $y_n > \frac{1}{2}$, for $n \geq 1$.

Therefore, $\lim_{n\to\infty} \varepsilon_n = 0$ but $\lim_{n\to\infty} y_n = \frac{1}{2}$, so the Picard iteration is not (S,T)-stable.

In order to show the (S,T)-weak stability, we take an approximate sequence $\{Sy_n\}$ of Sx_n. Then, there exists a decreasing sequence of nonnegative numbers $\{\eta_n\}$ converging to some $\eta \geq 0$ for $\eta \to \infty$ such that

$$|Sx_n - Sy_n| \leq \eta_n, \ \ n \geq k.$$

Then, $-\eta_n \leq Sx_n - Sy_n \leq \eta_n$ and results that $0 \leq y_n \leq x_n + \eta_n, \ \ n \geq k$.

Since $x_n = 0$, for $n \geq 2$, we obtain $0 \leq y_n \leq \eta_n, \ \ n \geq k_1 = \max\{2,k\}$.

We can choose $\{\eta_n\}$ such that $\eta_n \leq \frac{1}{2}$, $n \geq k_1$ and therefore $0 \leq y_n \leq \frac{1}{2}$, $\forall n \geq k_1$. So, $Ty_n = 0$ and results that $\varepsilon_n = |y_{n+1} - Ty_n| = |y_{n+1}| = y_{n+1}$.

Now, it is obvious that $\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} y_n = 0$, so the iteration $\{Sy_n\}$ is (S,T)-weakly stable.

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