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The refinement and generalization of the double Cosnita-Turtoiu inequality with one parameter

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ABSTRACT.
In this short note, we give the refinement and generalization of the double Cosnita-Turtoiu inequality with one parameter by Gerretsen's inequality, Euler's inequality and an equivalent form of fundamental triangle inequality.

1. INTRODUCTION AND MAIN RESULTS

For a given triangle ABC, let a, b, c be the side-lengths, h_a, h_b, h_c the altitudes, s the semi-perimeter, $\triangle$ the area, R the circumradius and r the inradius, respectively. Moreover, we will customarily use the cyclic sum symbol, that is:

$$\sum f(a) = f(a) + f(b) + f(c),$$

$$\sum f(b,c) = f(a,b) + f(b,c) + f(c,a)$$

and

$$\prod f(a) = f(a)f(b)f(c)$$

etc.

In 1965, C. Cosnita and F. Turtoiu (see [2, pp. 66, Theorem 6.22]) built the following so-called **Cosnita-Turtoiu** inequality.

$$6 \le \sum \frac{h_a + r}{h_a - r} \tag{1.1}$$

In 2003, Tian [5] considered the upper-bound of **Cosnita-Turtoiu** inequality and get the result as follows.

$$\sum \frac{h_a + r}{h_a - r} < 7 \tag{1.2}$$

In fact, Zhang [6] obtained the following result in 1998.

$$\frac{19}{3} - \frac{2r}{3R} \le \sum \frac{h_a + r}{h_a - r} \le 7 - \frac{2r}{R} \tag{1.3}$$

It's easy to find that the right hand of inequality (1.3) is better than inequality (1.2).

And in 1999, Chu [3] generalized **Cosnita-Turtoiu** inequality with one parameter.

$$\frac{3(3+\lambda)}{3-\lambda} \le \sum \frac{h_a + \lambda r}{h_a - \lambda r} (\lambda \le 2) \tag{1.4}$$

In this paper, we establish the following result.

Theorem 1.1. *In $\triangle ABC$, we have*

$$\frac{6+\lambda}{2-\lambda} - \frac{4\lambda^2}{(\lambda-2)(\lambda-3)} \cdot \frac{r}{R} \ge \sum \frac{h_a + \lambda r}{h_a - \lambda r}$$
$$\ge \begin{cases} \dfrac{3(3+\lambda)}{3-\lambda} + \dfrac{2\lambda^2}{(4-\lambda)(3-\lambda)(2-\lambda)} \cdot \left(1 - \dfrac{2r}{R}\right) \left(\lambda < \dfrac{7-\sqrt{17}}{2}\right), \\ \dfrac{3(3+\lambda)}{3-\lambda} + \dfrac{16(\lambda-1)}{(3-\lambda)^3} \cdot \left(1 - \dfrac{2r}{R}\right) \left(\dfrac{7-\sqrt{17}}{2} \le \lambda \le 2\right). \end{cases} \tag{1.5}$$

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2. Preliminary results

Lemma 2.1. *In $\triangle ABC$, we have*

$$\sum \frac{h_a+\lambda r}{h_a-\lambda r} = \frac{(6+\lambda)(2-\lambda)s^2+\lambda^2(6\lambda-4)Rr-\lambda^2r^2}{(\lambda-2)^2s^2-\lambda^2(2\lambda-4)Rr+\lambda^2r^2} \tag{2.6}$$

Proof. Utilizing the known identities

$$h_a=\frac{2\triangle}{a},\ h_b=\frac{2\triangle}{b},\ h_c=\frac{2\triangle}{c} \text{ and } \triangle = rs,$$

we can get

$$\begin{aligned}\sum \frac{h_a+\lambda r}{h_a-\lambda r} &= \sum \frac{\frac{2rs}{a}+\lambda r}{\frac{2rs}{a}-\lambda r} = \sum \frac{2s+\lambda a}{2s-\lambda a} = \sum \frac{(1+\lambda)a+b+c}{(1-\lambda)a+b+c} \\ &= \frac{\sum[(1+\lambda)a+b+c][(1-\lambda)b+c+a][(1-\lambda)c+a+b]}{\prod[(1-\lambda)a+b+c]} \\ &= \frac{(3-\lambda)(\sum a)^3-\lambda^2\sum a\sum bc+3\lambda^3\prod a}{(1-\lambda)(\sum a)^3+\lambda^2\sum a\sum bc-\lambda^3\prod a}.\end{aligned} \tag{2.7}$$

With known identities [4, pp. 52]:

$$a+b+c=2s,\ ab+bc+ca=s^2+4Rr+r^2,\ abc=4Rrs,$$

together with (2.7), we obtain identity (2.6) immediately. □

Lemma 2.2. ([1]) *For any triangle ABC, the following inequalities hold true:*

$$\frac{1}{4}\delta(4-\delta)^3 \le \frac{s^2}{R^2} \le \frac{1}{4}(2-\delta)(2+\delta)^3, \tag{2.8}$$

where $\delta = 1-\sqrt{1-\frac{2r}{R}} \in (0,1]$. Furthermore, the equality holds in left (or right) inequality of (2.8) *if and only if the triangle is isosceles.*

Lemma 2.3. *If $0<\delta\le 1$ and $\lambda < \frac{7-\sqrt{17}}{2}$, then*

$$(\lambda-1)\delta+\lambda^2-7\lambda+8\ge 0. \tag{2.9}$$

Proof. Define the function

$$f(\delta)=(\lambda-1)\delta+\lambda^2-7\lambda+8,\ \delta\in(0,1] \quad \left(\lambda<\frac{7-\sqrt{17}}{2}\right).$$

It's easy to see $f(\delta)$ is a linear function with respect to δ. Hence, inequality (2.9) holds if and only if $f(0)\ge 0$ and $f(1)\ge 0$.

(i) For $\lambda < \frac{7-\sqrt{17}}{2} < \frac{7+\sqrt{17}}{2}$, it's easy to find

$$f(0)=\lambda^2-7\lambda+8=\left(\lambda-\frac{7-\sqrt{17}}{2}\right)\left(\lambda-\frac{7+\sqrt{17}}{2}\right)>0.$$

(ii) For $\lambda<\frac{7-\sqrt{17}}{2}<3-\sqrt{2}<3+\sqrt{2}$, we can get

$$f(1)=\lambda^2-6\lambda+7=[\lambda-(3-\sqrt{2})][\lambda-(3+\sqrt{2})]>0.$$

From (i) and (ii), we can get inequality (2.9) immediately. □

3. The proof of theorem 1.1

Proof. We prove Theorem 1.1 with three steps.

(i) First, we prove the left hand of inequality (1.5).
By Lemma 2.1 and $\lambda<2$, the left hand of inequality (1.5) is equivalent to

$$\frac{(6+\lambda)(2-\lambda)s^2+\lambda^2(6\lambda-4)Rr-\lambda^2r^2}{(\lambda-2)^2s^2-\lambda^2(2\lambda-4)Rr+\lambda^2r^2} \le \frac{6+\lambda}{2-\lambda}-\frac{4\lambda^2}{(\lambda-2)(\lambda-3)}\cdot\frac{r}{R}$$

or

$$\frac{4\lambda^2 r}{(\lambda-2)(\lambda-3)[(\lambda-2)^2s^2-\lambda^2(2\lambda-4)Rr+\lambda^2r^2]} \cdot[-(\lambda-2)^2s^2-(\lambda-2)(\lambda-3)(\lambda-4)R^2 +2(\lambda^3-2\lambda^2-\lambda+3)Rr-\lambda^2r^2]\geq 0. \tag{3.10}$$

And inequality (3.10) is equivalent to

$$(2-\lambda)^2(4R^2+4Rr+3r^2-s^2)+(2-\lambda)(\lambda^2-3\lambda+4)(R-2r)^2 + 2[(2-\lambda)^3+(2-\lambda)+1](R-2r)r\geq 0. \tag{3.11}$$

From $\lambda<2$, **Gerretsen**'s inequality ([2, pp. 50, Theorem 5.8])

$$s^2\leq 4R^2+4Rr+3r^2$$

and **Euler**'s inequality ([2, pp. 48, Theorem 5.1])

$$R\geq 2r,$$

we can conclude that inequality (3.11) holds.

(ii) Second, we prove the following inequality.

$$\sum\frac{h_a+\lambda r}{h_a-\lambda r}\geq\frac{3(3+\lambda)}{3-\lambda)}+\frac{2\lambda^2(1-\frac{2r}{R})}{(4-\lambda)(3-\lambda)(2-\lambda)}\left(\lambda<\frac{7-\sqrt{17}}{2}\right) \tag{3.12}$$

By Lemma 2.1, inequality (3.12) is equivalent to

$$\frac{(6+\lambda)(2-\lambda)s^2+\lambda^2(6\lambda-4)Rr-\lambda^2r^2}{(\lambda-2)^2s^2-\lambda^2(2\lambda-4)Rr+\lambda^2r^2} \geq\frac{3(3+\lambda)}{3-\lambda)}+\frac{2\lambda^2}{(4-\lambda)(3-\lambda)(2-\lambda)}\cdot\left(1-\frac{2r}{R}\right)\left(\lambda<\frac{7-\sqrt{17}}{2}\right). \tag{3.13}$$

For $\lambda<\frac{7-\sqrt{17}}{2}$, inequality (3.13) is equivalent to

$$(2-\lambda)^2[(3-\lambda)R+2r](s^2-16Rr+5r^2)+4(\lambda^2-5\lambda+5)(R-2r)r^2\geq 0. \tag{3.14}$$

From $\lambda<\frac{7-\sqrt{17}}{2}$ and Lemma 2.2, we can get

$$\begin{aligned}&(2-\lambda)^2[(3-\lambda)R+2r](s^2-16Rr+5r^2)+4(\lambda^2-5\lambda+5)(R-2r)r^2\\ \geq&(2-\lambda)^2[(3-\lambda)R+2r]\left[\frac{1}{4}\delta(4-\delta)^3R^2-16Rr+5r^2\right]\\ &+4(\lambda^2-5\lambda+5)(R-2r)r^2\\ =&\delta^2(\delta-1)^2(4-\delta-\lambda)[(\lambda-1)\delta+\lambda^2-7\lambda+8]R^3\end{aligned} \tag{3.15}$$

For $\lambda<\frac{7-\sqrt{17}}{2}$ and $0<\delta\leq 1$, we can easily find

$$4-\delta-\lambda>0. \tag{3.16}$$

From inequality (3.16), $0<\delta\leq 1$ and Lemma 2.3, we can get

$$\delta^2(\delta-1)^2(4-\delta-\lambda)[(\lambda-1)\delta+\lambda^2-7\lambda+8]R^3\geq 0. \tag{3.17}$$

Inequality (3.14) follows from inequalities (3.15) and (3.17) immediately, hence, inequality (3.12) holds.

(iii) Third, we prove the following inequality.

$$\sum\frac{h_a+\lambda r}{h_a-\lambda r}\geq\frac{3(3+\lambda)}{3-\lambda)}+\frac{16(\lambda-1)}{(3-\lambda)^3}\cdot\left(1-\frac{2r}{R}\right)\left(\frac{7-\sqrt{17}}{2}\leq\lambda\leq 2\right) \tag{3.18}$$

With Lemma 2.1, inequality (3.18) is equivalent to

$$\frac{(6+\lambda)(2-\lambda)s^2+\lambda^2(6\lambda-4)Rr-\lambda^2r^2}{(\lambda-2)^2s^2-\lambda^2(2\lambda-4)Rr+\lambda^2r^2} \geq\frac{3(3+\lambda)}{3-\lambda)}+\frac{16(\lambda-1)}{(3-\lambda)^3}\cdot\left(1-\frac{2r}{R}\right)\left(\frac{7-\sqrt{17}}{2}\leq\lambda\leq 2\right) \tag{3.19}$$

For $\lambda \le 2$, inequality (3.19) is equivalent to

$$(2-\lambda)[(\lambda^2-3\lambda+4)^2(R-2r)+2\lambda^2(\lambda-3)^2r]s^2+\lambda^2(14\lambda^3-92\lambda^2 \\ +222\lambda-184)R^2r-\lambda^2(\lambda^3+32\lambda^2-115\lambda+110)Rr^2+16(\lambda-1)r^3\ge 0. \tag{3.20}$$

From $\lambda \le 2$ and Lemma 2.2, we can get

$$\begin{aligned}&(2-\lambda)[(\lambda^2-3\lambda+4)^2(R-2r)+2\lambda^2(\lambda-3)^2r]s^2+\lambda^2(14\lambda^3-92\lambda^2\\&+222\lambda-184)R^2r-\lambda^2(\lambda^3+32\lambda^2-115\lambda+110)Rr^2+16(\lambda-1)r^3\\ \ge&(2-\lambda)[(\lambda^2-3\lambda+4)^2(R-2r)+2\lambda^2(\lambda-3)^2r]\cdot\frac{1}{4}\delta(4-\delta)^3R^2\\&+\lambda^2(14\lambda^3-92\lambda^2+222\lambda-184)R^2r-\lambda^2(\lambda^3+32\lambda^2-115\lambda+110)Rr^2\\&+16(\lambda-1)r^3\\=&2\delta(\delta-1)^2(4-\delta-\lambda)[2(\lambda-1)\delta+\lambda^2-7\lambda+8]^2R^3.\end{aligned} \tag{3.21}$$

From inequality (3.16) and $0<\delta\le 1$, we can get

$$2\delta(\delta-1)^2(4-\delta-\lambda)[2(\lambda-1)\delta+\lambda^2-7\lambda+8]^2R^3\ge 0. \tag{3.22}$$

Inequality (3.20) follows from inequalities (3.21) and (3.22) immediately. Thus, Inequality (3.18) holds.

From the three steps above, the proof of Theorem 1.1 is completed. □

Remark 3.1. From the proof of Theorem 1.1, we can find that inequality (3.18) holds for $\lambda \le 2$, but inequality (3.12) is better than inequality (3.18) when $\lambda < \frac{7-\sqrt{17}}{2}$, because

$$\begin{aligned}&\frac{2\lambda^2}{(4-\lambda)(3-\lambda)(2-\lambda)}-\frac{16(\lambda-1)}{(3-\lambda)^3}=\frac{2(\lambda^2-7\lambda+8)^2}{(3-\lambda)^3(4-\lambda)(2-\lambda)}>0\\ \Longrightarrow&\frac{2\lambda^2}{(4-\lambda)(3-\lambda)(2-\lambda)}>\frac{16(\lambda-1)}{(3-\lambda)^3}.\end{aligned}$$

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