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Metric relations in the plane obtained by using complex numbers

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ABSTRACT.
The main purpose of this paper is to obtain some important metric relations in the Euclidean plane by using complex numbers. In the first Section we prove the Lagrange's formula (Theorem 1.1) and we derive a generalization of the well-known Stewart's relation (Theorem 1.3). Section 2 is devoted to Zarantonello's inequality (Theorem 2.4) and to its connection to the problem of finding the group of plane isometries (Theorem 2.5).

1. LAGRANGE'S THEOREM AND SOME CONSEQUENCES

The method of using complex numbers in Geometry is a strong one. It can be used to solve some important classes of problems, especially involving distances, angles and orthogonality, as well as collinearity. These problems are called metric problems. There is a vast literature on this topic, we only mention here the books [1], [4], [5], [6], and [7]. In this section we will prove the well-known Lagrange's theorem in the Euclidean plane by using complex numbers and their properties. This result is also proved in book [1] (see Theorem 1, Section 4.11, pp. 141-148), where the proof uses the real product of two complex numbers.

Theorem 1.1 (Lagrange's Theorem). *Let n be a positive integer, let $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $\alpha_1 + \cdots + \alpha_n = 1$. Then for each $z \in \mathbb{C}$ the following relation holds:*

$$\sum_{k=1}^{n} \alpha_k |z - z_k|^2 = |z - \alpha_1 z_1 - \cdots - \alpha_n z_n|^2 + \sum_{k=1}^{n} \alpha_k |z_k - \alpha_1 z_1 - \cdots - \alpha_n z_n|^2.$$

Proof. Using the properties of the absolute value, we obtain

$$\sum_{k=1}^{n} \alpha_k |z - z_k|^2 = \sum_{k=1}^{n} \alpha_k (z - z_k)(\overline{z} - \overline{z}_k) = \sum_{k=1}^{n} \alpha_k (|z|^2 - z\overline{z}_k - \overline{z} z_k + |z_k|^2)$$
$$= |z|^2 - z \sum_{k=1}^{n} \alpha_k \overline{z}_k - \overline{z} \sum_{k=1}^{n} \alpha_k z_k + \sum_{k=1}^{n} \alpha_k |z_k|^2. \tag{1.1}$$

On the other hand

$$|z - \alpha_1 z_1 - \cdots - \alpha_n z_n|^2 + \sum_{k=1}^{n} \alpha_k |z_k - \alpha_1 z_1 - \cdots - \alpha_n z_n|^2$$
$$= (z - \alpha_1 z_1 - \cdots - \alpha_n z_n)(\overline{z} - \alpha_1 \overline{z}_1 - \cdots - \alpha_n \overline{z}_n)$$
$$+ \sum_{k=1}^{n} \alpha_k (z_k - \alpha_1 z_1 - \cdots - \alpha_n z_n)(\overline{z}_k - \alpha_1 \overline{z}_1 - \cdots - \alpha_n \overline{z}_n)$$
$$= |z|^2 - z \sum_{k=1}^{n} \alpha_k \overline{z}_k - \overline{z} \sum_{k=1}^{n} \alpha_k z_k + 2(\alpha_1 z_1 + \cdots + \alpha_n z_n)(\alpha_1 \overline{z}_1 + \cdots + \alpha_n \overline{z}_n)$$
$$+ \sum_{k=1}^{n} \alpha_k |z_k|^2 - \sum_{k=1}^{n} \alpha_k z_k (\alpha_1 \overline{z}_1 + \cdots + \alpha_n \overline{z}_n) - \sum_{k=1}^{n} \alpha_k \overline{z}_k (\alpha_1 z_1 + \cdots + \alpha_n z_n)$$
$$= |z|^2 - z \sum_{k=1}^{n} \alpha_k \overline{z}_k - \overline{z} \sum_{k=1}^{n} \alpha_k z_k + \sum_{k=1}^{n} \alpha_k |z_k|^2. \tag{1.2}$$

From (1.1) and (1.2) we obtain the relation to be proven. □

Remark 1.1. From a geometric point of view, we can rewrite Theorem 1.1 in the following metric form:

Let n be a positive integer, let $P_1, \ldots, P_n$ be n points in the plane and let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ be real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$. Then for each point P in the plane, the following relation holds:

$$\sum_{k=1}^{n} \alpha_k PP_k^2 = PG^2 + \sum_{k=1}^{n} \alpha_k P_k G^2,$$

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where G is the barycenter of the set $\{P_1, \ldots, P_n\}$ with respect to the weights $\alpha_1, \ldots, \alpha_n$. In particular, if $\alpha_1 = \cdots = \alpha_n = \frac{1}{n}$, then G is the centroid of the polygon with vertices $P_1, \ldots, P_n$.

Theorem 1.2 (Stewart's Theorem). *Let $z_1, z_2 \in \mathbb{C}$ and $a \in \mathbb{R}$. Then for each $z \in \mathbb{C}$ the following relation holds:*

$$|z - az_1 - (1-a)z_2|^2 = a|z - z_1|^2 + (1-a)|z - z_2|^2 - a(1-a)|z_1 - z_2|^2.$$

Proof. Using the properties of the absolute value, the left hand side of the equation can be written as

$$\begin{aligned}
|z - az_1 - (1-a)z_2|^2 &= |z + az - az - az_1 - (1-a)z_2|^2 \\
&= |a(z - z_1) + (1-a)(z - z_2)|^2 \\
&= [a(z - z_1) + (1-a)(z - z_2)][a(\overline{z} - \overline{z}_1) + (1-a)(\overline{z} - \overline{z}_2)] \\
&= a^2|z - z_1|^2 + (1-a)^2|z - z_2|^2 \\
&\quad + a(1-a)[(z - z_1)(\overline{z} - \overline{z}_2) + (z - z_2)(\overline{z} - \overline{z}_1)] \\
&= a^2|z - z_1|^2 + (1-a)^2|z - z_2|^2 \\
&\quad + a(1-a)[|z|^2 - z_1\overline{z} - z\overline{z}_2 + z_1\overline{z}_2 + |z|^2 - z\overline{z}_1 - z_2\overline{z} + z_2\overline{z}_1].
\end{aligned}$$

We get

$$\begin{aligned}
|z - az_1 - (1-a)z_2|^2 &= a^2|z - z_1|^2 + (1-a)^2|z - z_2|^2 \\
&\quad + a(1-a)[|z - z_1|^2 + |z - z_2|^2 - |z_1|^2 - |z_2|^2 + z_1\overline{z}_2 + z_2\overline{z}_1] \\
&= a|z - z_1|^2 + (1-a)|z - z_2|^2 - a(1-a)|z_1 - z_2|^2.
\end{aligned}$$

□

Remark 1.2. From a geometric point of view, we can rewrite Theorem 1.2 in the following metric form:

Let P_1, P_2 be two points in the plane and let $a \in [0, 1]$. For each point P in the plane, the following relation holds:

$$PM^2 = aPP_1^2 + (1-a)PP_2^2 - a(1-a)P_1P_2^2,$$

where M is the point which divides the segment P_1P_2 into the ratio a, thus we have

$$a = \frac{MP_2}{P_1P_2}.$$

Replacing a, our relation becomes the well-known Stewart's relation (see [3, pp. 6]):

$$PP_1^2 \cdot MP_2 + PP_2^2 \cdot MP_1 = PM^2 \cdot P_1P_2 + MP_1 \cdot MP_2 \cdot P_1P_2.$$

This theorem is useful in computing the length of a cevian in a triangle when we know the ratio a.

Stewart's theorem can be generalized in the following way:

Theorem 1.3. *Let n be a positive integer, $n \geq 2$, $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $a_1 + \cdots + a_n = 1$. Then for each $z \in \mathbb{C}$ we have:*

$$\left| z - \sum_{k=1}^{n} a_k z_k \right|^2 = \sum_{k=1}^{n} a_k |z - z_k|^2 - \sum_{1 \leq k < l \leq n} a_k a_l |z_k - z_l|^2.$$

Proof. The relation we have to prove is equivalent to

$$\left(z - \sum_{k=1}^{n} a_k z_k\right)\left(\overline{z} - \sum_{k=1}^{n} a_k \overline{z}_k\right) = \sum_{k=1}^{n} a_k (z - z_k)(\overline{z} - \overline{z}_k) - \sum_{1 \leq k < l \leq n} a_k a_l (z_k - z_l)(\overline{z}_k - \overline{z}_l)$$

and we obtain

$$\begin{aligned}
&z\overline{z} - \sum_{k=1}^{n} a_k z\overline{z}_k - \sum_{k=1}^{n} a_k z_k \overline{z} + \sum_{1 \leq k, l \leq n} a_k a_l z_k \overline{z}_l \\
&= \sum_{k=1}^{n} a_k (z\overline{z} - z\overline{z}_k - z_k\overline{z} + z_k\overline{z}_k) - \sum_{1 \leq k < l \leq n} a_k a_l (z_k\overline{z}_k - z_k\overline{z}_l - z_l\overline{z}_k + z_l\overline{z}_l).
\end{aligned}$$

It suffices to prove that

$$\sum_{1 \leq k < l \leq n} a_k a_l (z_k\overline{z}_k + z_l\overline{z}_l) + \sum_{k=1}^{n} a_k^2 z_k \overline{z}_k = \sum_{k=1}^{n} a_k z_k \overline{z}_k,$$

which is equivalent to

$$\sum_{k=1}^{n} a_k (1 - a_k) z_k \overline{z}_k = \sum_{1 \leq k < l \leq n} a_k a_l (z_k\overline{z}_k + z_l\overline{z}_l),$$

and we get

$$\sum_{k=1}^{n} a_k(a_1+\cdots+a_{k-1}+a_{k+1}+\cdots+a_n)z_k\overline{z}_k = \sum_{1\le k<l\le n} a_k a_l(z_k\overline{z}_k + z_l\overline{z}_l),$$

which is true, so the theorem is proved. □

Remark 1.3. From a geometric point of view, we can rewrite Theorem 1.3 in the following metric form:

Let n be a positive integer, $n \ge 2$, let $P_1, P_2, \dots, P_n$ be n points in the plane and let $a_1, a_2, \dots, a_n \in \mathbb{R}$ be real numbers such that $a_1 + \cdots + a_n = 1$. Then for each point P in the plane, the following relation holds:

$$PG^2 = \sum_{k=1}^{n} a_k PP_k^2 - \sum_{1\le k<l\le n} a_k a_l P_kP_l^2,$$

where G is the barycenter of the set $\{P_1, \dots, P_n\}$ with respect to the weights $a_1, \dots, a_n$. If $a_1, \dots, a_n \ge 0$, then G is a point in the convex envelope of the polygon with vertices $P_1, \dots, P_n$.

Theorem 1.3 shows a way of computing the distance from a point in the plane to a point in the convex envelope of a finite set of points in the plane.

Theorem 1.3 has some interesting consequences:

Corollary 1.1. *If n is a positive integer, $n \ge 2$, $z_1, z_2, \dots, z_n \in \mathbb{C}$, then for each $z \in \mathbb{C}$ the following relation holds:*

$$\left| z - \frac{z_1+z_2+\cdots+z_n}{n} \right|^2 = \frac{1}{n}\sum_{k=1}^{n} |z-z_k|^2 - \frac{1}{n^2}\sum_{1\le k<l\le n} |z_k - z_l|^2.$$

Proof. In Theorem 1.3, we take $a_1 = a_2 = \cdots = a_n = \dfrac{1}{n}$. □

Remark 1.4. Corollary 1.1 can be rewritten in the following metric form:

Let n be a positive integer, $n \ge 2$ and let $P_1, \dots, P_n$ be n points in the plane. Then for each point P in the plane, the following relation holds:

$$PG^2 = \frac{1}{n}\sum_{k=1}^{n} PP_k^2 - \frac{1}{n^2}\sum_{1\le k<l\le n} P_kP_l^2,$$

where G is the centroid of the polygon with vertices $P_1, P_2, \dots, P_n$.

Corollary 1.2. *If n is a positive integer, $n \ge 2$, $z_1, z_2, \dots, z_n \in \mathbb{C}$, then*

$$\sum_{k=1}^{n} \left| z_k - \frac{z_1+z_2+\cdots+z_n}{n} \right|^2 = \sum_{k=1}^{n} |z_k|^2 - n\left| \frac{z_1+z_2+\cdots+z_n}{n} \right|^2.$$

Proof. In Corollary 1.1 we take $z = 0$, then we take $z = \dfrac{z_1+z_2+\cdots+z_n}{n}$ and we obtain the equality we wanted to prove. □

Remark 1.5. From a geometric point of view, we can rewrite Corollary 1.2 in the following metric form:

Let n be a positive integer, $n \ge 2$ and let $P_1, \dots, P_n$ be n points in the plane. Then the following relation holds:

$$\sum_{k=1}^{n} P_kG^2 = \sum_{k=1}^{n} OP_k^2 - nOG^2,$$

where G is the centroid of the polygon with vertices $P_1, P_2, \dots, P_n$ and O is the origin of the complex plane.

Corollary 1.3. *If n is a positive integer, $n \ge 2$, $z_1, z_2, \dots, z_n \in \mathbb{C}$, then*

$$\sum_{1\le k<l\le n} |z_k - z_l|^2 = n\sum_{k=1}^{n} |z_k|^2 - n^2\left| \frac{1}{n}\sum_{k=1}^{n} z_k \right|^2.$$

Proof. In Corollary 1.1, we take $z = 0$. □

Remark 1.6. Corollary 1.3 can be rewritten as:

Let n be a positive integer, $n \ge 2$ and let $P_1, \dots, P_n$ be n points in the plane. Then

$$\sum_{1\le k<l\le n} P_kP_l^2 = n\sum_{k=1}^{n} OP_k^2 - n^2 OG^2,$$

where G is the centroid of the polygon with vertices $P_1, P_2, \dots, P_n$ and O is the origin of the complex plane.

Corollary 1.4. *If n is a positive integer, $n \geq 2$, $z_1, z_2, \ldots, z_n \in \mathbb{C}$, $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n \in \mathbb{R}$ are real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n = 1$, then the following relation holds:*

$$|b_1 z_1 + \cdots + b_n z_n - a_1 z_1 - \cdots - a_n z_n|^2 = -\sum_{1 \leq k < l \leq n} (a_k - b_k)(a_l - b_l)|z_k - z_l|^2.$$

Proof. Using Theorem 1.3, we get

$$\begin{aligned}
|b_1 z_1 + \cdots + b_n z_n - a_1 z_1 - \cdots - a_n z_n|^2 &= a_1|z_1 - b_1 z_1 - \cdots - b_n z_n|^2 + \ldots \\
+a_n|z_n - b_1 z_1 - \cdots - b_n z_n|^2 - \sum_{1 \leq k < l \leq n} a_k a_l |z_k - z_l|^2 &= a_1(b_1|z_1 - z_1|^2 + \ldots \\
+b_n|z_1 - z_n|^2 - \sum_{1 \leq k < l \leq n} b_k b_l |z_k - z_l|^2) &+ \cdots + a_n(b_1|z_n - z_1|^2 + \ldots \\
+b_n|z_n - z_n|^2 - \sum_{1 \leq k < l \leq n} b_k b_l |z_k - z_l|^2) &- \sum_{1 \leq k < l \leq n} a_k a_l |z_k - z_l|^2.
\end{aligned}$$

Since $a_1 + a_2 + \cdots + a_n = 1$, we obtain

$$\begin{aligned}
|b_1 z_1 + \cdots + b_n z_n - a_1 z_1 - \cdots - a_n z_n|^2 &= \sum_{1 \leq k, l \leq n} a_k b_l |z_k - z_l|^2 - \sum_{1 \leq k < l \leq n} a_k a_l |z_k - z_l|^2 \\
- \sum_{1 \leq k < l \leq n} b_k b_l |z_k - z_l|^2 &= -\sum_{1 \leq k < l \leq n} (a_k - b_k)(a_l - b_l)|z_k - z_l|^2.
\end{aligned}$$

□

Remark 1.7. Corollary 1.4 can be rewritten as:

Let n be a positive integer, $n \geq 2$, let $P_1, \ldots, P_n$ be n points in the plane and let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n \in \mathbb{R}$ be real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n = 1$. Then the following relation holds:

$$MN^2 = -\sum_{1 \leq k < l \leq n} (a_k - b_k)(a_l - b_l) P_k P_l^2,$$

where M and N are the barycenters of the set $\{P_1, \ldots, P_n\}$ with respect to the weights $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively. If $a_1, \ldots, a_n \geq 0$ and $b_1, \ldots, b_n \geq 0$, then M and N are points in the convex envelope of the polygon with vertices $P_1, \ldots, P_n$.

Corollary 1.4 is useful in computing the distance between two points in the affine envelope of a polygon. This result extends the result contained in [1, Theorem 2, pp.117], where the formula is obtained for a triangle and the distance is expressed in terms of the barycentric coordinates of the points. As a nice application, if we consider the origin of the complex plane at the circumcenter of triangle ABC, then the Nagel point of the triangle has the complex coordinate $z_N = (1 - \frac{\alpha}{s})a + (1 - \frac{\beta}{s})b + (1 - \frac{\gamma}{s})c$, and we get the formula $ON = R - 2r$ (for more details see [1, pp.113] or paper [2]). Here α, β, γ represent the sides lengths, s the semi-perimeter, and a, b, c the complex coordinates of the vertices of triangle ABC.

The next result, also a consequence of Theorem 1.3, solves in $\mathbb{C}$ an extreme problem which in the real case was solved by E. Laguerre.

Corollary 1.5. *Let n be a positive integer, $n \geq 2$, $z_1, z_2, \ldots, z_n \in \mathbb{C}$, $z_0 = \dfrac{z_1 + \cdots + z_n}{n}$ and let $c = \dfrac{1}{n^2} \sum_{1 \leq k < l \leq n} |z_k - z_l|^2$. Then for each $k \in \{1, 2, \ldots, n\}$ the following relation holds:*

$$|z_k - z_0| \leq \sqrt{(n-1)c}.$$

Proof. In Corollary 1.1 we take $z = z_0$ and we obtain that

$$|z_1 - z_0|^2 + \cdots + |z_n - z_0|^2 = nc.$$

Take $k = 1$. By applying Cauchy-Schwarz inequality, we have

$$(|z_2 - z_0| + |z_3 - z_0| + \cdots + |z_n - z_0|)^2 \leq (n-1)(|z_2 - z_0|^2 + \cdots + |z_n - z_0|^2).$$

Using the absolute value inequality, we obtain that

$$|z_2 + z_3 + \cdots + z_n - (n-1)z_0|^2 \leq (n-1)(|z_2 - z_0|^2 + \cdots + |z_n - z_0|^2),$$

which is equivalent to

$$|nz_0 - z_1 - (n-1)z_0|^2 \leq (n-1)(|z_2 - z_0|^2 + \cdots + |z_n - z_0|^2),$$

and we get

$$n|z_1 - z_0|^2 \leq (n-1)(|z_1 - z_0|^2 + |z_2 - z_0|^2 + \cdots + |z_n - z_0|^2) = n(n-1)c.$$

Thus we obtain

$$|z_1 - z_0| \leq \sqrt{(n-1)c}.$$

□

Remark 1.8. In the real case, E. Laguerre's result says that:
Let n be a positive integer, $x_1, x_2, \dots, x_n \in \mathbb{R}$,

$$a = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad b = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}.$$

Then for each $k \in \{1, 2, \dots, n\}$ the following relation holds:

$$|x_k - a| \le \sqrt{(n-1)(b-a^2)}.$$

For $n = 1$, the relation holds. If $n \ge 2$, then from Corollary 1.5 we get

$$|x_k - a| \le \sqrt{(n-1)c}.$$

On the other hand $b - a^2 = c$ and thus the result in Corollary 1.5 is a generalization of Laguerre's theorem.

Remark 1.9. From a geometric point of view, we can rewrite Corollary 1.5 in the following metric form:
Let n be a positive integer, $n \ge 2$, let $P_1, P_2, \dots, P_n$ be n points in the plane, let G be the centroid of the polygon with vertices $P_1, P_2, \dots, P_n$ and let

$$c = \frac{1}{n^2} \sum_{1 \le k < l \le n} P_k P_l^2.$$

Then for each $k \in \{1, 2, \dots, n\}$ the following relation holds:

$$GP_k \le \sqrt{(n-1)c}.$$

2. Zarantonello's inequality and the isometries of the complex plane

In this section we will use the result in Theorem 1.3 in order to prove the so-called Zarantonello's inequality in the complex plane. This inequality was proved in [8] for mappings in Hilbert spaces. It will give us an useful instrument to describe the plane isometries.

Theorem 2.4 (Zarantonello's inequality). *Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that*

$$|f(z) - f(w)| \le |z - w|,$$

for each $z, w \in \mathbb{C}$. Then for any positive integer n, $n \ge 2$, for any real numbers $a_1, a_2, \dots, a_n \ge 0$ with $a_1 + a_2 + \cdots + a_n = 1$ and for any $z_1, z_2, \dots, z_n \in \mathbb{C}$, the following relation holds:

$$|f(a_1 z_1 + \cdots + a_n z_n) - a_1 f(z_1) - \cdots - a_n f(z_n)|^2 \le \sum_{1 \le k < l \le n} a_k a_l (|z_k - z_l|^2 - |f(z_k) - f(z_l)|^2).$$

Proof. By applying Theorem 1.3, we obtain

$$\begin{aligned} &|f(a_1 z_1 + \cdots + a_n z_n) - a_1 f(z_1) - \cdots - a_n f(z_n)|^2 \\ &= a_1 |f(a_1 z_1 + \cdots + a_n z_n) - f(z_1)|^2 + \dots \\ &+ a_n |f(a_1 z_1 + \cdots + a_n z_n) - f(z_n)|^2 - \sum_{1 \le k < l \le n} a_k a_l |f(z_k) - f(z_l)|^2. \end{aligned} \tag{2.3}$$

Using the contraction condition for f, we get

$$\begin{aligned} &a_1 |f(a_1 z_1 + \cdots + a_n z_n) - f(z_1)|^2 + \cdots + a_n |f(a_1 z_1 + \cdots + a_n z_n) - f(z_n)|^2 \\ &\le a_1 |z_1 - a_1 z_1 - \cdots - a_n z_n|^2 + \cdots + a_n |z_n - a_1 z_1 - \cdots - a_n z_n|^2. \end{aligned} \tag{2.4}$$

Taking in Theorem 1.3 $z = a_1 z_1 + \cdots + a_n z_n$, the following relation holds:

$$a_1 |z_1 - a_1 z_1 - \cdots - a_n z_n|^2 + \cdots + a_n |z_n - a_1 z_1 - \cdots - a_n z_n|^2 = \sum_{1 \le k < l \le n} a_k a_l |z_k - z_l|^2,$$

and together with relations (2.3) and (2.4) we obtain the inequality we wanted to prove. □

Corollary 2.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that*

$$|f(z) - f(w)| \le |z - w|,$$

and let $z_1, z_2, \dots, z_n \in \mathbb{C}$ be fixed complex numbers. If $|f(z_k) - f(z_l)| = |z_k - z_l|$, for $k, l = 1, 2, ..., n$, $k \ne l$, then for any real numbers $a_1, a_2, \dots, a_n \ge 0$ with $a_1 + a_2 + \cdots + a_n = 1$, we have:

$$f(a_1 z_1 + \cdots + a_n z_n) = a_1 f(z_1) + \cdots + a_n f(z_n).$$

Proof. Indeed, we have $|z_k - z_l|^2 - |f(z_k) - f(z_l)|^2 = 0$ and $a_k a_l \ge 0$ for $k, l = 1, 2, ..., n$, $k \ne l$. From Zarantonello's inequality it follows $|f(a_1 z_1 + \cdots + a_n z_n) - a_1 f(z_1) - \cdots - a_n f(z_n)|^2 = 0$, hence the conclusion. □

The result contained in the previous Corollary shows that any function $f : \mathbb{C} \to \mathbb{C}$ satisfying the contraction condition $|f(z) - f(w)| \leq |z - w|$, for any $z, w \in \mathbb{C}$ and preserving all distances in the set $\{z_1, z_2, \ldots, z_n\}$ is affine on the convex envelope of this set.

Recall that the function $f : \mathbb{C} \to \mathbb{C}$ is an *isometry*, if $|f(z) - f(w)| = |z - w|$, for each $z, w \in \mathbb{C}$. That is an isometry preserves the distances. A similar argument as in the proof of the previous Corollary shows that the following result holds:

For any positive integer n, $n \geq 2$, for any real numbers $a_1, a_2, \ldots, a_n \geq 0$ with $a_1 + a_2 + \cdots + a_n = 1$ and for any $z_1, z_2, \ldots, z_n \in \mathbb{C}$ the following relation holds:

$$f(a_1 z_1 + \cdots + a_n z_n) = a_1 f(z_1) + \cdots + a_n f(z_n).$$

This simple remark is useful to determine all isometries of the Euclidean plane.

Theorem 2.5. *A function $f : \mathbb{C} \to \mathbb{C}$ is an isometry if and only if there exists $\alpha, \beta \in \mathbb{C}$, $|\alpha| = 1$ such that $f(z) = \alpha z + \beta$ or $f(z) = \alpha\overline{z} + \beta$, for each $z \in \mathbb{C}$.*

Proof. If $f(z) = \alpha z + \beta$ or $f(z) = \alpha\overline{z} + \beta$, $|\alpha| = 1$, it is easy to check that f is an isometry.

Conversely, let us define $g : \mathbb{C} \to \mathbb{C}$, $g(z) = f(z) - f(0)$. If $f : \mathbb{C} \to \mathbb{C}$ is an isometry, i.e. $|f(z) - f(w)| = |z - w|$, for each $z, w \in \mathbb{C}$, then the function $g : \mathbb{C} \to \mathbb{C}$, defined by $g(z) = f(z) - f(0)$ is additive, i.e.

$$g(z + w) = g(z) + g(w),$$

for each $z, w \in \mathbb{C}$.

Indeed, we notice that g is also an isometry and we obtain

$$g\left(\frac{z + w}{2}\right) = \frac{g(z) + g(w)}{2},$$

for each $z, w \in \mathbb{C}$. Since $g(0) = 0$, we get

$$g\left(\frac{z}{2}\right) = \frac{g(z)}{2},$$

for each $z \in \mathbb{C}$.

Thus, we have proved that

$$g(z + w) = 2g\left(\frac{z + w}{2}\right) = 2\frac{g(z) + g(w)}{2} = g(z) + g(w),$$

for each $z, w \in \mathbb{C}$.

It is easy to prove that $g(t \cdot z) = t \cdot g(z)$, for each $t \in \mathbb{R}$ and $z \in \mathbb{C}$.

Let $z = x + iy$. Then we have

$$g(z) = g(x + iy) = g(x) + g(iy) = \alpha \cdot x + \gamma \cdot y,$$

where $\alpha = g(1)$ and $\gamma = g(i)$. From the relations $|g(1) - g(0)| = |1 - 0|$ and $|g(i) - g(0)| = |i - 0|$ we obtain that $|\alpha| = |\gamma| = 1$, and from the relation $|g(1 \pm i) - g(0)| = |1 \pm i| = \sqrt{2}$ we obtain that $|\alpha \pm \gamma| = \sqrt{2}$.

Thus, $\gamma = i \cdot \alpha$ or $\gamma = -i \cdot \alpha$, and we get $g(z) = \alpha x + \alpha i y = \alpha z$ or $g(z) = \alpha x - i\alpha y = \alpha\overline{z}$. □

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