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Some results on discrete Banach lattices

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ABSTRACT.
We establish some characterizations of discrete Banach lattices and obtain some interesting consequences.

1. INTRODUCTION AND NOTATION

Recall that a nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements.

Discrete Banach lattices played a fundamental role in the operator theory on Banach lattices. In fact, they were used to study the domination problem for many class of operators. For examples, in [7, Theorem 1] and [3, Theorem 2.5], this notion was used to establish necessary and sufficient conditions under which the domination problem for compact operators admits a positive solution. Also, the same notion was utilized in [4, Theorem 2.11] to prove necessary and sufficient conditions for the domination problem for the class of AM-compact operators on Banach lattices.

In [8], Wnuk established a characterization of discrete Banach lattices with order continuous norms. The objective of this paper is to give other characterizations of discrete Banach lattices. More precisely, we will prove that a Banach lattice E with an order continuous norm, is discrete if and only if its topological dual E' is discrete. Also, we will establish that the topological dual E', of a Banach lattice E, is discrete and of order continuous norm if and only if its topological bidual E'' is discrete. Finally, we will show that a Banach lattice E is reflexive and discrete if and only if E''' is discrete.

To state our results, we need to fix some notations and recall some definitions. A vector lattice E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace F of a vector lattice E is said to be a sublattice if for every pair of elements a, b of F the supremum of a and b taken in E belongs to F. A subset B of a vector lattice E is said to be solid if it follows from $|y| \leq |x|$ with $x \in B$ and $y \in E$ that $y \in B$. An order ideal of E is a solid subspace. Let E be a vector lattice, for each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E : x \leq z \leq y\}$ is called an order interval. A subset of E is said to be order bounded if it is included in some order interval. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowing with the dual norm, is also a Banach lattice. We refer to [9] for unexplained terminology on Banach lattices theory.

Also, a vector lattice equipped with a vector topology is said to be a locally convex solid lattice if zero admits a fundamental system of convex and solid neighborhoods.

The topology τ of a locally convex solid vector lattice is said to be Lebesgue if each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, converges to 0 for the topology τ, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing and $\inf(x_\alpha) = 0$.

If E' is the topological dual of E, the absolute weak topology $|\sigma|(E, E')$ is the locally convex solid topology on E generated by the family of lattice seminorms $\{P_f : f \in E'\}$ where $P_f(x) = |f|(|x|)$ for each $x \in E$. Similarly, $|\sigma|(E', E)$ is the locally convex solid topology on E' generated by the family of lattice seminorms $\{P_x : x \in E\}$ where $P_x(f) = |f|(|x|)$ for each $f \in E'$. For more information about locally convex solid topologies, we refer the reader to the book of Aliprantis and Burkinshaw [1].

2. MAIN RESULTS

Recall that if (E, E^*) is a dual system and A is a family of bounded subsets of E for the weak topology $\sigma(E, E^*)$, we define the $A-$topology on E^* as the topology of uniform convergence on elements of A . It is a locally convex topology generated by the family of semi-norms $\{P_A : A \in A\}$ where $P_A(x^*) = \sup\{|\langle x, x^*\rangle| : x \in A\}$ for each $x^* \in E^*$.

Similarly, if B^* is the family of bounded subsets of E^* for the topology $\sigma(E^*, E)$, we define the B^*-topology on E as the topology of uniform convergence on elements of B^* .

If $T : E \to F$ is a weakly continuous linear mapping relatively to dual systems (E, E^*) and (F, F^*), the dual mapping $T^* : F^* \to E^*$ is defined by $\langle x, T^*(y^*)\rangle = \langle T(x), y^*\rangle$ for each $y^* \in F^*$ and for each $x \in E$.

The following result is called the duality theorem of Grothendieck that we can find in [5, Theorem 3, p. 51].

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Theorem 2.1. *Let (E, E^*) and (F, F^*) be two dual systems and let $T : E \to F$ be a weakly continuous linear mapping. Let A and B^* be sets of weakly bounded subsets of E and F^* defining topologies of uniform convergence on E^* and F. Then the following assertions are equivalent:*

(1) *For each $A \in A$, $T(A)$ is precompact for the B^*-topology.*
(2) *For each $B^* \in B^*$, $T(B^*)$ is precompact for the A-topology.*

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. An operator $T : E \longrightarrow F$ is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous.

Let us recall that if an operator $T : E \longrightarrow F$ between two Banach lattices is positive (i.e. $T(x) \geq 0$ in F whenever $x \geq 0$ in E), then its dual operator $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$.

As a consequence, we obtain the following corollary:

Corollary 2.1. *Let E and F be two Banach lattices and $T : E \to F$ an operator. Then T maps order intervals of E onto precompact subsets of F for $|\sigma|(F, F')$ if and only if T' maps order intervals of F' onto precompact subsets of E' for $|\sigma|(E', E)$.*

Proof. To see this, we apply Theorem 2.1 with $A = \{[-x, x] : x \in E^+\}$ and $B^* = \{[-y', y'] : y' \in (F')^+\}$ and we note that the A-topology is $|\sigma|(E', E)$ and the B^*-topology is $|\sigma|(F, F')$. □

Recall that a vector lattice is said to be order complete if every nonempty subset that is bounded from above has a supremum.

The following theorem gives a characterization of Banach lattices whose topological duals are discrete.

Theorem 2.2. *Let E be a Banach lattice, then the following assertions are equivalent:*

(1) *The topological dual E' is discrete.*
(2) *Each order interval of E' is compact for $|\sigma|(E', E)$.*
(3) *Each order interval of E' is precompact for $|\sigma|(E', E)$.*
(4) *Each order bounded subset of E' is precompact for $|\sigma|(E', E)$.*
(5) *For each $x' \in (E')^+$, $[-x', x']$ is precompact for $|\sigma|(E', E)$.*
(6) *For each $x \in E^+$, $[-x, x]$ is precompact for $|\sigma|(E, E')$.*
(7) *Each order interval of E is precompact for $|\sigma|(E, E')$.*
(8) *Each order bounded subset of E is precompact for $|\sigma|(E, E')$.*
(9) *For each positive operator T from E into E, $T([0, x])$ is precompact for the topology $|\sigma|(F, F')$ for each $x \in E^+$.*

Proof. (1) ⇔ (2) Since E' is order complete and the topology $|\sigma|(E', E)$ is Lebesgue (see Theorem 19.6 of [1]), it follows form Corollary 21.13 of [1] that (1) ⇔ (2).

(2) ⇒ (3) Obvious.

(3) ⇒ (2) Since E' is complete for $|\sigma|(E', E)$ (See Theorem 19.7 of [1]), then each order interval of E' is complete for $|\sigma|(E', E)$. Hence (3) ⇒ (2)

(3) ⇔ (4) ⇔ (5) Obvious.

(5) ⇔ (6) It is just Corollary 2.1 with $T = Id_E$ where Id_E is the identity operator of E.

(6) ⇔ (7) ⇔ (8) Obvious.

(9) ⇒ (1) Let T be a positive operator from E into E. If T' is the adjoint operator from E' into E' defined by $T'(f)(x) = f(T(x))$ for each $f \in E'$ and for each $x \in E$, it is clear that T' is positive and for each $f \in (E')^+$, we have $T'([0, f]) \subset [0, T'(f)]$ where $(E')^+ = \{f \in E' : 0 \leq f\}$. Since E' is discrete and order complete, and since the weak absolute topology $|\sigma|(E', E)$ is Lebesgue, it follows from Corollary 21.13 of [1] that each order interval $[0, T'(f)]$ is compact for $|\sigma|(E', E)$, and then $[0, T'(f)]$ is precompact for $|\sigma|(E', E)$. Hence, $T'([0, f])$ is precompact for $|\sigma|(E', E)$. Now, an application of Theorem 1.3 of [2] implies that for each $x \in E^+$, $T([0, x])$ is precompact for $|\sigma|(E, E')$. This proves the result.

(1) ⇒ (9) Assume that each positive operator T from E into E satisfies $T([0, x])$ is precompact for the topology $|\sigma|(E, E')$ for each $x \in E^+$. Then the identity operator of E satisfies this property. Hence, for each $x \in E^+$, the order interval $[0, x]$ is precompact for $|\sigma|(E, E')$. On the other hand, Theorem 1.3 of [2] implies that for each $f \in F'$, $[0, f]$ is precompact for $|\sigma|(E', E)$. Since $[0, f]$ is complete for the weak absolute topology $|\sigma|(E', E)$, it follows from Theorem 21.12 of [1] that $[0, f]$ is compact for $|\sigma|(E', E)$. Finally, the result comes from Corollary 21.13 of [1]. □

Recall that a Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$.
The Banach lattice E is an AL-space if its topological dual E' is an AM-space. For examples, the Banach lattice l^1 is an AL-space and the Banach lattice l^∞ is an AM-space.

To give a consequence of Theorem 2.2, recall that the lattice operations in a Banach lattice E are weakly sequentially continuous if the sequence $(|x_n|)$ converges to 0 for the weak topology $\sigma(E, E')$ whenever the sequence (x_n)

converges to 0 for $\sigma(E, E')$. For examples, the lattice operations of an AM-space are weakly sequentially continuous but the lattice operations of the Banach lattice L^2 are not.

An interesting consequence of our Theorem 2.3 and Theorem 1.4 of [2], is given by the following corollary:

Corollary 2.2. *Let E be a Banach lattice. If E' is discrete, then the lattice operations of E are weakly sequentially continuous.*

Proof. By Theorem 2.2, for each $x \in E^+$, the order interval $[0, x]$ is precompact for the weak absolute topology $|\sigma|(E, E')$. Now, by applying Theorem 1.4 of [2], we deduce that for each sequence $(x_n) \subset E$, which converges to 0 weakly, the sequence $(|x_n|)$ converges also to 0 weakly. □

Proposition 2.1. *Let N be a vector sublattice of the normed vector lattice E. Then $|\sigma|(E, E')$ induces the topology $|\sigma|(N, N')$ on N.*

Proof. Let (x_α) be a generalized sequence of N such that $x_\alpha \to 0$ holds in N for $|\sigma|(E, E')$ i.e. we have $\lim |\varphi|(|x_\alpha|) = 0$ for every $\varphi \in E'$. This is equivalent to say that $\lim \varphi(|x_\alpha|) = 0$ for every $\varphi \in (E')^+$.

If $\varphi \in (N')^+$ i.e., φ is a continuous positive linear form on N, then it follows from Proposition 5.6 of [6] that φ has a norm preserving positive (linear) extension $\varphi^\sim$ to E, and hence $\lim \varphi(|x_\alpha|) = \lim \varphi^\sim(|x_\alpha|) = 0$. Then $\lim |\varphi|(|x_\alpha|) = 0$ for every $\varphi \in N'$ i.e., $x_\alpha \to 0$ for $|\sigma|(N, N')$.

Conversely, assume that $x_\alpha \to 0$ holds in N for $|\sigma|(N, N')$. If $\varphi \in (E')^+$, then $\varphi_{|_N} \in (N')^+$ where $\varphi_{|_N}$ denotes the restriction of φ on N . Then $\lim \varphi(|x_\alpha|) = \lim \varphi_{|_N}(|x_\alpha|) = 0$ for every $\varphi \in (E')^+$ and hence $\lim |\varphi|(|x_\alpha|) = 0$. It follows that $x_\alpha \to 0$ for $|\sigma|(E, E')$ and this proves the result. □

The following property gives the discreteness of the topological dual of closed sublattices.

Proposition 2.2. *Let N be a closed sublattice of the Banach lattice E. If E' is discrete, then N' is also discrete.*

Proof. Let A be an order bounded subset of N. Then A is an order bounded subset of E. Since E' is discrete, it follows from Theorem 2.2 that A is precompact for $|\sigma|(E, E')$. Now, by Proposition 2.1, we deduce that A is precompact for $|\sigma|(N, N')$. This holds for an arbitrary order bounded subset A of N. Finally, another application of Theorem 2.2 shows that N' is discrete. □

A vector lattice is said to be order σ-complete if every nonempty countable subset that is bounded from above has a supremum.

Now, we give the relation between the discreteness of a Banach lattice and the discreteness of its topological dual.

Theorem 2.3. *Let E be a Banach lattice, then the following assertions are equivalent:*

(1) *E is discrete and its norm is order continuous.*
(2) *Each order interval of E is compact for the norm.*
(3) *Each order interval of E is compact for $|\sigma|(E, E')$.*
(4) *E is σ-order complete and its topological dual E' is discrete.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) They are just Corollary 21.13 of [1].

(3) $\Rightarrow$ (4) Assume that each order interval of E is compact for $|\sigma|(E, E')$. Then each order interval of E is precompact for $|\sigma|(E, E')$. Now, Theorem 2.2 implies that E' is discrete. On the other hand, it follows from (1) that E is order complete, hence E is σ-order complete.

(4) $\Rightarrow$ (2) First, we note that E has an order continuous norm. Otherwise, E would contain a closed sublattice order and topologically isomorphic to the Banach lattice l^∞ (see proof of Lemma 1 of [8]). Since E' is discrete, it follows from Proposition 2.2 that the topological dual $(l^\infty)'$ is also discrete. But, this is impossible, so E has an order continuous norm.

On the other hand, since E' is discrete, it follows from Theorem 2.2 that each order interval of E is precompact for $|\sigma|(E, E')$. As E has an order continuous norm, it follows from Theorem 1.4 of [2] that $|\sigma|(E, E')$ and the norm topology induce the same topology on each order interval of E. Then we deduce that each order interval of E is precompact for the norm, and hence each order interval of E is compact for the norm. This proves the result. □

As a consequence, we obtain a sufficient condition under which the discreteness of a Banach lattice is equivalent to the discreteness of its topological dual.

Corollary 2.3. *Let E be a Banach lattice with an order continuous norm. Then E is discrete if and only if E' is discrete.*

Proof. Since the Banach lattice E has an order continuous norm, then E is order complete. Hence E is order σ-complete and the result follows from Theorem 2.3. □

Corollary 2.4. *Let E be an order σ-complete Banach lattice. Then E is discrete and its norm is order continuous if and only if E' is discrete.*

The following consequence gives the relation between the discreteness of the topological dual and the topological bidual of a Banach lattice.

Corollary 2.5. *Let E be a Banach lattice. Then the following assertions are equivalent:*

(1) E' *is discrete and its norm is order continuous.*
(2) *each order interval of* E' *is compact for the norm.*
(3) *each order interval of* E' *is compact for* $|\sigma|(E', E'')$.
(4) E'' *is discrete.*

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ They are just Corollary 21.13 of [1].

$(1) \Leftrightarrow (4)$ Since the Banach lattice E' is order complete, then E' is order σ-complete. If we replace E by E' in Theorem 2.3, we obtain the result. □

Recall from Theorem 5.16 of [6], that a Banach lattice E is reflexive if and only if the norms of its topological dual E' and of its topological bidual E'' are order continuous. Also, if E'' has an order continuous norm then so has E.

For a Banach lattice E, the following consequence gives other relations on the discreteness of E, E', E'' and E'''.

Corollary 2.6. *Let* E *be a Banach lattice. Then the following assertions are equivalent:*

(1) E *is reflexive and discrete.*
(2) E' *is reflexive and discrete.*
(3) E'' *is discrete and its norm is order continuous.*
(4) E''' *is discrete.*

Proof. It follows from Corollary 2.5 and Theorem 5.16 of [6] that E''' is discrete if and only if E'' is discrete and its norm is order continuous, if and only if E' is discrete and the norms of E' and E'' are order continuous, if and only if E' is discrete and E is reflexive, if and only if E is reflexive and discrete. □

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