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A generalization of Radon's Inequality

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ABSTRACT.
In this paper we prove a generalization of Radon's Inequality and give some applications.

1. INTRODUCTION

Let $\mathbb{N}$ be positive integers, $\mathbb{N} = \{1, 2, \dots\}$. The inequality from (1.1) is called, in literature, Bergström's Inequality (see [1], [2], [3] or [9]).

Theorem 1.1. *If $n \in \mathbb{N}$, $x_k \in \mathbb{R}$ and $y_k > 0$, $k \in \{1, 2, \dots, n\}$, then*

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \dots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{y_1 + y_2 + \dots + y_n}, \tag{1.1}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n}$.

In [11], J. Radon proves the inequality from Theorem 1.2, called Radon's Inequality.

Theorem 1.2. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \dots, n\}$ and $m \geq 0$, then*

$$\frac{x_1^{m+1}}{y_1^m} + \frac{x_2^{m+1}}{y_2^m} + \dots + \frac{x_n^{m+1}}{y_n^m} \geq \frac{(x_1 + x_2 + \dots + x_n)^{m+1}}{(y_1 + y_2 + \dots + y_n)^m}. \tag{1.2}$$

2. MAIN RESULTS AND APPLICATIONS

In this section, we give two proofs for a generalization of Radon's Inequality.

Theorem 2.3. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \dots, n\}$, $m \geq 0$ and $p \geq 1$, then*

$$\frac{x_1^{m+p}}{y_1^m} + \frac{x_2^{m+p}}{y_2^m} + \dots + \frac{x_n^{m+p}}{y_n^m} \geq \frac{\left(x_1 y_1^{p-1} + x_2 y_2^{p-1} + \dots + x_n y_n^{p-1}\right)^{m+p}}{\left(y_1^p + y_2^p + \dots + y_n^p\right)^{m+p-1}}, \tag{2.3}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \dots = \frac{x_n}{y_n}$.

First proof. In the following, we denote $U_n(p) = x_1 y_1^{p-1} + x_2 y_2^{p-1} + \dots + x_n y_n^{p-1}$ and $Y_n(p) = y_1^p + y_2^p + \dots + y_n^p$. The left hand side of (2.3), can be written as

$$\sum_{k=1}^{n} \frac{x_k^{m+p}}{y_k^m} = Y_n(p) \sum_{k=1}^{n} \frac{y_k^p}{Y_n(p)} \left(\frac{x_k}{y_k}\right)^{m+p}. \tag{2.4}$$

The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^{m+p}$, $x \in [0, \infty)$, is convex on $[0, \infty)$, so, for any $n \in \mathbb{N}$, $q_1, q_2, \dots, q_n \in (0, 1]$ with $q_1 + q_2 + \dots + q_n = 1$ and $z_1, z_2, \dots, z_n \in [0, \infty)$, we have that $\sum\limits_{k=1}^{n} q_k f(z_k) \geq f\left(\sum\limits_{k=1}^{n} q_k z_k\right)$, in view of Jensen inequality. In the inequality above, the equality holds if and only if $z_1 = z_2 = \dots = z_n$. We consider $q_k = \frac{y_k^p}{Y_n(p)}$, $z_k = \frac{x_k}{y_k}$, $k \in \{1, 2, \dots, n\}$ and then

$$\sum_{k=1}^{n} \frac{y_k^p}{Y_n(p)} \left(\frac{x_k}{y_k}\right)^{m+p} \geq \left(\sum_{k=1}^{n} \frac{y_k^p}{Y_n(p)} \frac{x_k}{y_k}\right)^{m+p},$$

which we get

$$\sum_{k=1}^{n} \frac{y_k^p}{Y_n(p)} \left(\frac{x_k}{y_k}\right)^{m+p} \geq \frac{(U_n(p))^{m+p}}{(Y_n(p))^{m+p}}. \tag{2.5}$$

From (2.4) and (2.5), the inequality (2.3) follows. □

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Second proof. The inequality (2.3) is equivalent to

$$\sum_{k=1}^{n}\left(\frac{x_k}{U_n(p)}\right)^{m+p}\frac{(Y_n(p))^{m+p-1}}{y_k^m}\geq 1. \tag{2.6}$$

If $k\in\{1,2,\ldots,n\}$ then by applying Bernoulli's Inequality, we have that

$$\left(\frac{x_k}{U_n(p)}\right)^{m+p}\frac{(Y_n(p))^{m+p-1}}{y_k^m}=\frac{y_k^p}{Y_n(p)}\left(\frac{x_k}{U_n(p)}\,\frac{Y_n(p)}{y_k}\right)^{m+p}$$
$$=\frac{y_k^p}{Y_n(p)}\left(1+\frac{x_kY_n(p)-y_kU_n(p)}{y_kU_n(p)}\right)^{m+p}\geq\frac{y_k^p}{Y_n(p)}\left(1+(m+p)\frac{x_kY_n(p)-y_kU_n(p)}{y_kU_n(p)}\right),$$

so

$$\left(\frac{x_k}{U_n(p)}\right)^{m+p}\frac{(Y_n(p))^{m+p-1}}{y_k^m}\geq\frac{y_k^p}{Y_n(p)}+(m+p)\left(\frac{x_ky_k^{p-1}}{U_n(p)}-\frac{y_k^p}{Y_n(p)}\right). \tag{2.7}$$

Take $k\in\{1,2,\ldots,n\}$ in (2.7) and summing, we obtain relation (2.6). □

Theorem 2.4. *Radon's Inequality and inequality* (2.3) *are equivalent.*

Proof. If we consider p=1 in inequality (2.3), we obtain Radon's Inequality. By simultaneously substituting in (1.2), m with $m+p-1$, x_k with $x_ky_k^{p-1}$ and y_k with y_k^p, where $k\in\{1,2,\ldots,n\}$, then from the classical Radon's Inequality it results (2.3). □

Theorem 2.5. *The following inequalities are equivalent:*

(i) *Hölder's Inequality,*
(ii) *Bernoulli's Inequality,*
(iii) *Radon's Inequality*
and
(iv) *inequality* (2.3).

Proof. It results from Theorem 2.4 and Proposition 3.2 from [6]. □

Application 2.1. If $a,b,c>0$, prove that

$$\frac{a^5}{b^2}+\frac{b^5}{c^2}+\frac{c^5}{a^2}\geq\frac{(ab^2+bc^2+ca^2)^5}{(a^3+b^3+c^3)^4}. \tag{2.8}$$

Solution. If $n=3$, $m=2$ and $p=3$, from (2.3) the inequality (2.8) follows.

Application 2.2. If $a,b,c>0$ with $a^3+b^3+c^3=1$, then the inequality

$$\frac{(b+c)^4}{a}+\frac{(c+a)^4}{b}+\frac{(a+b)^4}{c}\geq[(a+b+c)(a^2+b^2+c^2)-1]^4 \tag{2.9}$$

holds.

Solution. Taking inequality (2.3) into account, we have

$$\frac{(b+c)^4}{a}+\frac{(c+a)^4}{b}+\frac{(a+b)^4}{c}\geq\frac{((b+c)a^2+(c+a)b^2+(a+b)c^2)^4}{(a^3+b^3+c^3)^3}$$
$$=\frac{((a+b+c)(a^2+b^2+c^2)-(a^3+b^3+c^3))^4}{(a^3+b^3+c^3)^3}$$

and because $a^3+b^3+c^3=1$ we get the desired result.

Application 2.3. If $a,b,c>0$ with $a^9+b^9+c^9=1$, then

$$\frac{\sqrt[3]{a^{19}}}{\sqrt{b}}+\frac{\sqrt[3]{b^{19}}}{\sqrt{c}}+\frac{\sqrt[3]{c^{19}}}{\sqrt{a}}\geq\sqrt[6]{(a^2b^6+b^2c^6+c^2a^6)^{19}}. \tag{2.10}$$

Solution. Applying inequality (2.3) for $n=3$, $m=\frac{1}{6}$ and $p=3$, we have

$$\frac{\sqrt[3]{a^{19}}}{\sqrt{b}}+\frac{\sqrt[3]{b^{19}}}{\sqrt{c}}+\frac{\sqrt[3]{c^{19}}}{\sqrt{a}}=\frac{(a^2)^{\frac{1}{6}+3}}{(b^3)^{\frac{1}{6}}}+\frac{(b^2)^{\frac{1}{6}+3}}{(c^3)^{\frac{1}{6}}}+\frac{(c^2)^{\frac{1}{6}+3}}{(a^3)^{\frac{1}{6}}} \tag{2.11}$$
$$\geq\frac{(a^2b^6+b^2c^6+c^2a^6)^{\frac{19}{6}}}{(b^9+c^9+a^9)^{\frac{13}{6}}}$$

and because $a^9+b^9+c^9=1$, from the inequality above, (2.11) is obtained.

Application 2.4. If $n \in \mathbb{N}$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$, $m \geq 0$ and $p \geq 1$, then

$$\frac{1}{y_1^m} + \frac{1}{y_2^m} + \cdots + \frac{1}{y_n^m} \geq \frac{\left(y_1^{p-1} + y_2^{p-1} + \cdots + y_n^{p-1}\right)^{m+p}}{\left(y_1^p + y_2^p + \cdots + y_n^p\right)^{m+p-1}}. \tag{2.12}$$

Solution. In inequality (2.3) we consider $x_k = 1$, $k \in \{1, 2, \ldots, n\}$.

Application 2.5. If $n \in \mathbb{N}$, $u_k, y_k > 0$, $k \in \{1, 2, \ldots, n\}$, $m \geq 0$ and $p \geq 1$, the following inequality

$$\frac{\sum\limits_{k=1}^{n} u_k^{m+p} y_k^p}{\sum\limits_{k=1}^{n} y_k^p} \geq \left(\frac{\sum\limits_{k=1}^{n} u_k y_k^p}{\sum\limits_{k=1}^{n} y_k^p} \right)^{m+p} \tag{2.13}$$

holds.

Solution. In inequality (2.3) we choose $x_k = u_k y_k$, $k \in \{1, 2, \ldots, n\}$ and then we get the inequality (2.13).

Corollary 2.1. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$ and $m \geq 0$, then*

$$\frac{x_1^{m+2}}{y_1^m} + \frac{x_2^{m+2}}{y_2^m} + \cdots + \frac{x_n^{m+2}}{y_n^m} \geq \frac{(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^{m+2}}{(y_1^2 + y_2^2 + \cdots + y_n^2)^{m+1}}. \tag{2.14}$$

Proof. We consider $p = 2$ in inequality (2.3). □

Remark 2.1. If $m = 0$ in (2.14), then Cauchy-Schwarz's Inequality is obtained.

Application 2.6. If $a, b, c > 0$, the following inequality

$$\frac{a^3}{a+2b} + \frac{b^3}{b+2c} + \frac{c^3}{c+2a} \geq \frac{1}{81} \frac{(a+b+c)^6}{(a^2+b^2+c^2)^2} \tag{2.15}$$

holds.

Solution. In (2.14) we consider $m = 1$, $x_1 = a$, $x_2 = b$, $x_3 = c$, $y_1 = a + 2b$, $y_2 = b + 2c$, $y_3 = c + 2a$ and then we have that

$$\begin{aligned}\frac{a^3}{a+2b} + \frac{b^3}{b+2c} + \frac{c^3}{c+2a} &\geq \frac{(a(a+2b) + b(b+2c) + c(c+2a))^3}{((a+2b)^2 + (b+2c)^2 + (c+2a)^2)^2} \\ &= \frac{(a+b+c)^6}{(5(a^2+b^2+c^2) + 4(ab+bc+ca))^2}.\end{aligned}$$

By taking the fact that $ab + bc + ca \leq a^2 + b^2 + c^2$ into account in the inequality above, we obtain (2.15). The equality holds if and only if $\dfrac{a}{a+2b} = \dfrac{b}{b+2c} = \dfrac{c}{c+2a}$ which is equivalent with $a = b = c$.

Application 2.7. If $a, b, c > 0$, prove that

$$a^2\sqrt{\frac{a}{b+c}} + b^2\sqrt{\frac{b}{c+a}} + c^2\sqrt{\frac{c}{a+b}} \geq \frac{1}{\sqrt{2}} \frac{(ab+bc+ca)^2}{a^2+b^2+c^2} \sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}. \tag{2.16}$$

Solution. By applying the inequality (2.14) for $m = \dfrac{1}{2}$, we have that

$$\begin{aligned}a^2\sqrt{\frac{a}{b+c}} + b^2\sqrt{\frac{b}{c+a}} + c^2\sqrt{\frac{c}{a+b}} &= \frac{a^{\frac{5}{2}}}{(b+c)^{\frac{1}{2}}} + \frac{b^{\frac{5}{2}}}{(c+a)^{\frac{1}{2}}} + \frac{c^{\frac{5}{2}}}{(a+b)^{\frac{1}{2}}} \\ \geq \frac{(a(b+c) + b(c+a) + c(a+b))^{\frac{5}{2}}}{((b+c)^2 + (c+a)^2 + (a+b)^2)^{\frac{3}{2}}} &= \frac{(2(ab+bc+ca))^{\frac{5}{2}}}{(2(a^2+b^2+c^2+ab+bc+ca))^{\frac{3}{2}}}.\end{aligned}$$

But $ab + bc + ca \leq a^2 + b^2 + c^2$ and then, for the inequality above, (2.16) follows.

Application 2.8. If $a, b, c > 0$ with $a^2 + b^2 + c^2 = 1$, prove that

$$\frac{1}{a(b+c)^3} + \frac{1}{b(c+a)^3} + \frac{1}{c(a+b)^3} \geq \frac{27}{8}. \tag{2.17}$$

Solution. By applying the inequality (2.14) for $m = 1$, we have

$$\begin{aligned}\frac{1}{a(b+c)^3} + \frac{1}{b(c+a)^3} + \frac{1}{c(a+b)^3} &= \frac{\frac{1}{(b+c)^3}}{a} + \frac{\frac{1}{(c+a)^3}}{b} + \frac{\frac{1}{(a+b)^3}}{c} \\ &\geq \frac{\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)^3}{(a^2+b^2+c^2)^2}.\end{aligned}$$

Taking Nesbitt's Inequality into account $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$ and because $a^2+b^2+c^2=1$, from the inequality above results the inequality desired.

Application 2.9. If a, b, c are the length of a triangle and $a^2+b^2+c^2=1$, prove that

$$\frac{a^3}{-a+b+c}+\frac{b^3}{a-b+c}+\frac{c^3}{a+b-c}\geq\frac{(2(ab+bc+ca)-1)^3}{(2(ab+bc+ca)-3)^2}. \tag{2.18}$$

Solution. By using the inequality (2.14), we have

$$\begin{aligned}&\frac{a^3}{-a+b+c}+\frac{b^3}{a-b+c}+\frac{c^3}{a+b-c}\\ \geq\ &\frac{(a(-a+b+c)+b(a-b+c)+c(a+b-c))^3}{((-a+b+c)^2+(a-b+c)^2+(a+b-c)^2)^2}\\ =\ &\frac{(2(ab+bc+ca)-(a^2+b^2+c^2))^3}{(3(a^2+b^2+c^2)-2(ab+bc+ca))^2}\end{aligned}$$

and because $a^2+b^2+c^2=1$, we get the inequality (2.18).

Application 2.10. If $a, b\geq 0$, $a+b>0$, $x, y, z>0$, $x+y+z=s$ and $x^2+y^2+z^2=1$, then

$$\frac{x^4}{(as+by)^2}+\frac{y^4}{(as+bz)^2}+\frac{z^4}{(as+bx)^2}\geq\frac{1}{16}\frac{((2a+b)s^2-b)^4}{((3a^2+2ab)s^2+b^2)^3}. \tag{2.19}$$

Solution. Taking inequality (2.14) into account, we have

$$\begin{aligned}&\frac{x^4}{(as+by)^2}+\frac{y^4}{(as+bz)^2}+\frac{z^4}{(as+bx)^2}\\ \geq\ &\frac{(x(as+by)+y(as+bz)+z(as+bx))^4}{((as+by)^2+(as+bz)^2+(as+bx)^2)^3}\\ =\ &\frac{(as^2+b(xy+yz+zx))^4}{(3a^2s^2+2abs^2+b^2(x^2+y^2+z^2))^3}.\end{aligned}$$

But $xy+yz+zx=\frac{(x+y+z)^2-(x^2+y^2+z^2)}{2}$ and because $x^2+y^2+z^2=1$, from the inequality above (2.19) is obtained.

Corollary 2.2. *If $a, b\in\mathbb{R}$, $a<b$, $m\geq 0$, $p\geq 1$, $f, g:[a,b]\to[0,\infty)$ are integrable functions on $[a,b]$, $g(x)>0$ for any $x\in[a,b]$, then the following inequality*

$$\int_a^b\frac{(f(x))^{m+p}}{(g(x))^m}dx\geq\frac{\left(\int_a^b f(x)(g(x))^{p-1}dx\right)^{m+p}}{\left(\int_a^b (g(x))^p dx\right)^{m+p-1}} \tag{2.20}$$

holds.

Proof. Let $n\in\mathbb{N}$ and $x_k=a+k\frac{b-a}{n}$, $k\in\{0,1,\ldots,n\}$. By Theorem 2.3, we get that

$$\sum_{k=1}^n\frac{(f(x_k))^{m+p}}{(g(x_k))^m}\geq\frac{\left(\sum_{k=1}^n f(x_k)(g(x_k))^{p-1}\right)^{m+p}}{\left(\sum_{k=1}^n (g(x_k))^p\right)^{m+p-1}}.$$

It results that

$$\sigma\left(\frac{f^{m+p}}{g^m},\Delta_n,x_k\right)\geq\frac{\left(\sigma(fg^{p-1},\Delta_n,x_k)\right)^{m+p}}{(\sigma(g^p,\Delta_n,x_k))^{m+p-1}},$$

where $\sigma\left(\frac{f^{m+p}}{g^m},\Delta_n,x_k\right)$ is the corresponding Riemann sum of function $\frac{f^{m+p}}{g^m}$, of $\Delta_n=(x_0,x_1,\ldots,x_n)$ division and the intermediate x_k points. By passing to limit in inequality above, when n tends to infinity, the inequality (2.20) follows. □

REFERENCES

[1] Bechenbach, E. F. and Bellman, R., *Inequalities*, Springer, Berlin, Göttingen and Heidelberg, 1961
[2] Bellman, R., *Notes on Matrix Theory* - IV (An inequality Due to Bergström), Amer. Math. Montly, **62** (1955), 172-173
[3] Bergström, H., *A triangle inequality for matrices*, Den Elfte Skandinaviske Matematikerkongress, 1949, Trodheim, Johan Grundt Tanums Forlag, Oslo, 1952, 264-267
[4] Hardy, G., Littlewood, J. E. and Pólya, G., *Inequalities*, Cambridge University Press, 1934
[5] Panaitopol, L., *Consequences of Hölder's Inequality*, Gazeta Matematică **4** (2002), 145-147 (Romanian)
[6] Marinescu, D. Ş. and Schwarz, D., *Radon's Inequality*, Revista de Matematică din Timişoara **4** (2009), 3-7 (in Romanian)
[7] Mărghidanu, D., Diaz-Barrero, J. L. and Rădulescu, S., *New refinements of some classical inequalities*, Math. Inequal. and Applications **12** (2009), No. 3, 513-518
[8] Mitrinovič, D. S., Pečarić, J. E. and Fink, A. M., *Classical and New Inequalities*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1993
[9] Mitrinovič, D. S., *Analytic Inequalities*, Springer-Verlag, Berlin, 1970
[10] Pop, O. T., *About Bergström's Inequality*, J. Math. Inequalities **3** (2009), No. 2, 237-242
[11] Radon, J., *Theorie und Anwendungen der absolut additiven Mengenfunktionen*, Sitzungsber. Acad. Wissen. Wien **122** (1913), 1295-1438

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