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Certain aspects of some geometric inequalities

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ABSTRACT.
In this paper we prove some new inequalities for the triangle. We also improve Euler's Edwards and Weitzenböck inequalities.

1. Introduction and terminology

Among the well known geometric inequalities, we recall the famous inequality $R \geq 2r$ of Euler [6], the inequality $s \geq 3\sqrt{3}r$ of Edwards [5], and the inequality

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\Delta \tag{1.1}$$

of Weitzenböck [12]. The more general form

$$\Delta \leq \frac{\sqrt{3}}{4}\left(\frac{a^k + b^k + c^k}{3}\right)^{\frac{2}{k}} \ (k > 0) \tag{1.2}$$

of (1.1) appeared in [7], and (1.1) appeared again as a problem in the IMO in 1961 [4, pp. 30, 337] .

In this paper, we shall obtain several improvements of these inequalities.

In the following, we will use the following notations: $a, b, c-$ the lengths of the sides, h_a, h_b, h_c are the lengths of the altitudes, s is the semi-perimeter; R is the circumradius, r is the inradius, and Δ is the area of the triangle ABC.

2. Main results

Lemma 2.1. *If $x, y \geq 0$ and $\lambda \in [0,1]$, then the inequality*

$$\left(\frac{x+y}{2}\right)^2 \geq [(1-\lambda)x + \lambda y] \cdot [\lambda x + (1-\lambda)y] \geq xy \tag{2.1}$$

holds.

Proof. This lemma is proved in [1]. Here we will give another proof.

The lemma says that if $D_\lambda(x,y) = [(1-\lambda)x + \lambda y] \cdot [\lambda x + (1-\lambda)y]$, where $x, y \geq 0$ and $\lambda \in [0,1]$, then $D_{\frac{1}{2}} \geq D_\lambda \geq D_0$. But this trivially follows from the fact that $D_\lambda(x,y)$, being nothing but $\left(\frac{x+y}{2}\right)^2 - (1-2\lambda)^2\left(\frac{x-y}{2}\right)^2$, increases as λ increases from 0 to $\frac{1}{2}$ (and from $D_\lambda = D_{1-\lambda}$). □

Theorem 2.1. *If $x, y \geq 0$ and $\lambda \in [0,1]$ then the inequality*

$$\left(\frac{x+y+z}{3}\right)^3 \geq [(1-\lambda)x + \lambda y] \cdot [(1-\lambda)y + \lambda z] \cdot [(1-\lambda)z + \lambda x] \geq xyz \tag{2.2}$$

holds.

Proof. The inequality

$$\left(\frac{x+y+z}{3}\right)^3 \geq [(1-\lambda)x + \lambda y] \cdot [(1-\lambda)y + \lambda z] \cdot [(1-\lambda)z + \lambda x]$$

trivially follows from the arithmetic-geometric mean inequality.

Now, without loss of generality, let us suppose that $z = \min\{x, y, z\}$. Dividing by z^3 the inequality $[(1-\lambda)x + \lambda y] \cdot [(1-\lambda)y + \lambda z] \cdot [(1-\lambda)z + \lambda x] \geq xyz$ and writing $\frac{x}{z} = u$ and $\frac{y}{z} = v$, this inequality becomes

$$[(1-\lambda)u + \lambda v] \cdot [(1-\lambda)v + \lambda] \cdot [(1-\lambda) + \lambda u] \geq uv. \tag{2.3}$$

We prove the inequality

$$[(1-\lambda)v + \lambda] \cdot [(1-\lambda) + \lambda u] \geq (1-\lambda)v + \lambda u. \tag{2.4}$$

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It is easy to see that inequality (2.4) is equivalent to the inequality $uv+1\geq u+v$, so $(u-1)(v-1)\geq 0$, which is true, because $u=\dfrac{x}{z}\geq 1$ and $v=\dfrac{y}{z}\geq 1$.

Hence, combining lemma (2.1) and inequality (2.4), we have

$$[(1-\lambda)u+\lambda v]\cdot[(1-\lambda)v+\lambda]\cdot[(1-\lambda)+\lambda u]\geq[(1-\lambda)u+\lambda v]\cdot[(1-\lambda)v+\lambda u]\geq u\nu.$$

Consequently, inequality (2.3) is proved. □

If we consider the expression

$$E_\lambda(x,y,z)=[(1-\lambda)x+\lambda y]\cdot[(1-\lambda)y+\lambda z]\cdot[(1-\lambda)z+\lambda x],$$

then relation (2.2) becomes

$$x+y+z\geq 3\sqrt[3]{E_\lambda(x,y,z)}\geq 3\sqrt[3]{xyz}. \tag{2.5}$$

Corollary 2.1. *In any triangle ABC, there are the following inequalities:*

$$R\geq\frac{2}{3\sqrt[3]{E_\lambda\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c}\right)}}\geq 2r, \tag{2.6}$$

$$s\geq 3\sqrt{\frac{3E_\lambda(s-a,s-b,s-c)}{s}}\geq 3\sqrt{3}r \tag{2.7}$$

and

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha}\geq 3\sqrt[3]{E_\lambda\left(a^{2\alpha},b^{2\alpha},c^{2\alpha}\right)}\geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha \tag{2.8}$$

for all integers $n\geq 0$, *for all* $\alpha>0$ *and* $\lambda\in[0,1]$.

Proof. Using the substitutions $x=\frac{1}{h_a}, y=\frac{1}{h_b}$ and $z=\frac{1}{h_c}$ in inequality (2.5), we obtain

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\geq 3\sqrt[3]{E_\lambda\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c}\right)}\geq 3\sqrt[3]{\frac{1}{h_ah_bh_c}}. \tag{2.9}$$

In view of the equalities

$h_a=\dfrac{2\Delta}{a}, h_b=\dfrac{2\Delta}{b}, h_c=\dfrac{2\Delta}{c}, \Delta=\dfrac{abc}{4R}$ and taking into account the inequality $\dfrac{3\sqrt{3}}{2}R\geq s$ of Padoa [9] and Euler's inequality $R\geq 2r$, we have

$$\frac{3\sqrt{3}}{4}R^2\geq sr=\Delta. \tag{2.10}$$

Therefore, we have

$$3\sqrt[3]{\frac{1}{h_ah_bh_c}}=\frac{3}{2\Delta}\sqrt[3]{abc}=\frac{3}{2}\sqrt[3]{\frac{4R}{\Delta^2}}\geq\frac{2}{R}.$$

If we use the identity

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$$

and the inequality from above, then inequality (2.9) becomes

$$\frac{1}{r}\geq 3\sqrt[3]{E_\lambda\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c}\right)}\geq\frac{2}{R}. \tag{2.11}$$

Consequently, inequality (2.6) holds.

If in inequality (2.5) we take $x=s-a, y=s-b$ and $z=s-c$, then we deduce the inequality

$$s\geq 3\sqrt[3]{E_\lambda(s-a,s-b,s-c)}\geq 3\sqrt[3]{(s-a)(s-b)(s-c)}=3\sqrt[3]{sr^2},$$

so

$$s^3\geq 27\,E_\lambda(s-a,s-b,s-c)\geq 27\,sr^2,$$

which means that

$$s\geq 3\sqrt{\frac{3E_\lambda(s-a,s-b,s-c)}{s}}\geq 3\sqrt{3}r.$$

Making the substitutions $x=a^\alpha, y=b^\alpha$ and $z=c^\alpha$ in inequality (2.5), we obtain the following inequality:

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha}\geq 3\sqrt[3]{E_\lambda\left(a^{2\alpha},b^{2\alpha},c^{2\alpha}\right)}\geq 3\left[\sqrt[3]{(abc)^2}\right]^\alpha.$$

Applying the inequality $\sqrt[3]{a^2b^2c^2} \geq \frac{4\Delta}{\sqrt{3}}$ of Pólya-Szegö [10, 11] or of Carlitz-Leuenberger [3], we deduce

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq 3\sqrt[3]{E_\lambda\left(a^{2\alpha}, b^{2\alpha}, c^{2\alpha}\right)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha.$$

Thus, the statement is true. □

Remark 2.1. *For $\alpha = 1$ in inequality (2.8), we obtain*

$$a^2 + b^2 + c^2 \geq 3\sqrt[3]{E_\lambda\left(a^2, b^2, c^2\right)} \geq 4\sqrt{3}\Delta$$

which proves Weitzenböck's inequality

We consider, also, another expression, namely

$$\begin{aligned} F_\lambda(x, y)(n) &= \left[\left(1 + (1-2\lambda)^n\right)x + \left(1 - (1-2\lambda)^n\right)y\right] \cdot \\ &\quad \left[\left(1 - (1-2\lambda)^n\right)x + \left(1 + (1-2\lambda)^n\right)y\right], \end{aligned} \tag{2.12}$$

with $\lambda \in [0, 1]$, for any $x, y \geq 0$ and for all integers $n \geq 0$.

Theorem 2.2. *There are the following relations:*

$$F_\lambda\left((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y\right)(n) = F_\lambda(x, y)(n+1), \tag{2.13}$$

$$F_{\frac{1}{2}}(x, y) \geq F_\lambda(x, y)(n) \geq F_0(x, y) \tag{2.14}$$

and

$$F_\lambda(x, y)(n+1) \geq F_\lambda(x, y)(n) \tag{2.15}$$

for any $\lambda \in [0, 1]$, for any $x, y \geq 0$ and for all integers ≥ 0.

Proof. These all follow trivially from the fact that $F_\lambda(x, y)(n)$ is the expression

$$F_\lambda(x, y)(n) = (x+y)^2 - (1-2\lambda)^{2n}(x-y)^2.$$

Thus, relation (2.13) is obtained as follows:

$$\begin{aligned} F_\lambda\left((1-\lambda)x + \lambda y, \lambda x + (1-\lambda)y\right)(n) &= (x+y)^2 - (1-2\lambda)^{2n}(1-2\lambda)^2(x-y)^2 \\ &= (x+y)^2 - (1-2\lambda)^{2(n+1)}(x-y)^2 = F_\lambda(x, y)(n+1). \end{aligned}$$

Inequality (2.14) follows from the obvious fact that F_λ increases as λ increases from 0 to $\frac{1}{2}$, and from $F_0(x, y) = 4xy, F_{1/2}(x, y) = (x+y)^2$. Similarly, inequality (2.15) follows from the fact that F_λ, increases with n.
Thus, the proof of Theorem 2.2 is complete. □

Remark 2.2. *In fact, n does not have to be a natural number and can range over positive reals.*

Corollary 2.2. *There are the following inequalities:*

$$x + y \geq \sqrt{F_\lambda(x, y)(n)} \geq 2\sqrt{xy}; \tag{2.16}$$

$$x^2 + y^2 \geq \sqrt{F_\lambda\left(x^2, y^2\right)(n)} \geq 2xy; \tag{2.17}$$

$$x + y + z \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda(x, y)(n)} \geq \sqrt{xy} + \sqrt{yz} + \sqrt{zx}; \tag{2.18}$$

$$x^2 + y^2 + z^2 \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(x^2, y^2\right)(n)} \geq xy + yz + zx; \tag{2.19}$$

$$x^2 + y^2 + z^2 + xy + yz + zx \geq \frac{1}{2}\sum_{cyclic}F_\lambda(x, y)(n) \geq 2(xy + yz + zx) \tag{2.20}$$

and

$$(x+y)(y+z)(z+x) \geq \sqrt{\prod_{cyclic}F_\lambda(x, y)(n)} \geq 8xyz \tag{2.21}$$

for any $x, y, z \geq 0$, for all integers $n \geq 0$ and $\lambda \in [0, 1]$.

Proof. From Theorem 2.2, we easily deduce inequality (2.16). Using the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.16), we obtain inequality (2.17). Adding (2.16) to its analogues

$$y + z \geq \sqrt{F_\lambda(y, z)(n)} \geq 2\sqrt{yz} \text{ and } z + x \geq \sqrt{F_\lambda(z, x)(n)} \geq 2\sqrt{zx},$$

we obtain

$$x + y + z \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda(x, y)(n)} \geq \sqrt{xy} + \sqrt{yz} + \sqrt{zx}.$$

It is easy to see that, by making the substitutions $x \to x^2$ and $y \to y^2$ in inequality (2.18), we obtain inequality (2.19). Adding (2.14) to its analogues $(y+z)^2 \geq F_\lambda(y,z)(n) \geq 4yz$ and $(z+x)^2 \geq F_\lambda(z,x)(n) \geq 4zx$, we obtain inequality (2.20).

Multiplying (2.16) to its analogues, we deduce inequality (2.21). □

Lemma 2.2. *For any triangle ABC, the following inequality*

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq \frac{4\Delta}{R}, \tag{2.22}$$

holds.

Proof. We apply the arithmetic-geometric mean inequality and we find that

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 3\sqrt[3]{abc}.$$

Suffice it to show that

$$3\sqrt[3]{abc} \geq \frac{4\Delta}{R}. \tag{2.23}$$

Inequality (2.23) is equivalent to $27abc \geq \frac{64\Delta^3}{R^3}$, so $27 \cdot 4R\Delta \geq \frac{64\Delta^3}{R^3}$, which means that $27R^4 \geq 16\Delta^2$, which is true from inequality (2.10). □

Corollary 2.3. *In any triangle ABC, there are the following inequalities:*

$$R \geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_\lambda\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n)}} \geq 2r; \tag{2.24}$$

$$s \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda(s-a,s-b)(n)} \geq 3\sqrt{3}r \tag{2.25}$$

and

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(a^{2\alpha},b^{2\alpha}\right)(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha, \tag{2.26}$$

for all integers $n \geq 0$, for all $\alpha > 0$ and $\lambda \in [0,1]$.

Proof. Making the substitutions $x = \frac{1}{h_a}$, $y = \frac{1}{h_b}$ and $z = \frac{1}{h_c}$ in inequality (2.18), we obtain

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n)} \geq \frac{1}{\sqrt{h_ah_b}}+\frac{1}{\sqrt{h_bh_c}}+\frac{1}{\sqrt{h_ch_a}}. \tag{2.27}$$

In view of the equalities

$$h_a = \frac{2\Delta}{a}, h_b = \frac{2\Delta}{b} \text{ and } h_c = \frac{2\Delta}{c},$$

we have

$$\frac{1}{\sqrt{h_ah_b}}+\frac{1}{\sqrt{h_bh_c}}+\frac{1}{\sqrt{h_ch_a}} = \frac{1}{2\Delta}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right).$$

From Lemma 2.2, we deduce

$$\frac{1}{\sqrt{h_ah_b}}+\frac{1}{\sqrt{h_bh_c}}+\frac{1}{\sqrt{h_ch_a}} \geq \frac{2}{R}. \tag{2.28}$$

If we use the identity

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{1}{r}$$

and inequality (2.28), then inequality (2.27) becomes

$$\frac{1}{r} \geq \frac{1}{2}\sum_{cyclic}\sqrt{F_\lambda\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n)} \geq \frac{2}{R}.$$

Consequently, we obtain

$$R \geq \frac{4}{\sum\limits_{cyclic}\sqrt{F_\lambda\left(\frac{1}{h_a},\frac{1}{h_b}\right)(n)}} \geq 2r.$$

If in inequality (2.18) we take $x = s-a, y = s-b$ and $z = s-c$, then we deduce the inequality

$$s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (s-a, s-b)(n)}$$
$$\geq \sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)}. \tag{2.1}$$

But, we know the identity $\sum\limits_{cyclic} \sqrt{(s-a, s-b)} = \sum\limits_{cyclic} \sqrt{bc} \sin \frac{A}{2}$.

Using the arithmetic-geometric mean inequality, we obtain

$$\sum_{cyclic} \sqrt{bc} \sin \frac{A}{2} \geq 3 \sqrt[3]{abc \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = 3 \sqrt[3]{4R\Delta \cdot \frac{r}{4R}} =$$
$$3\sqrt[3]{\Delta r} = 3\sqrt[3]{sr^2} \geq 3\sqrt[3]{3\sqrt{3}r^3} = 3\sqrt{3}r.$$

Hence,

$$\sqrt{(s-a)(s-b)} + \sqrt{(s-b)(s-c)} + \sqrt{(s-c)(s-a)} \geq 3\sqrt{3}r, \tag{2.30}$$

which means, combining inequalities (2.1) and (2.30), that

$$s \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (s-a, s-b)(n)} \geq 3\sqrt{3}r.$$

Using the substitutions $x = a^\alpha, y = b^\alpha$, and $z = c^\alpha$ in inequality (2.18), we obtain the following inequality:

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (a^{2\alpha}, b^{2\alpha})(n)} \geq a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha. \tag{2.31}$$

Applying the arithmetic- geometric mean inequality and the inequality, $\sqrt[3]{a^2b^2c^2} \geq \frac{4\Delta}{\sqrt{3}}$ of Pólya-Szegö [10, 11] or of Carlitz-Leuenberger [3], we deduce

$$a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha \geq 3\sqrt[3]{(a^2b^2c^2)^\alpha} = 3\left(\sqrt[3]{a^2b^2c^2}\right)^\alpha \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha,$$

so

$$a^\alpha b^\alpha + b^\alpha c^\alpha + c^\alpha a^\alpha \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha. \tag{2.32}$$

Combining inequalities (2.31) and (2.32), we obtain the inequality

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (a^{2\alpha}, b^{2\alpha})(n)} \geq 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha.$$

The proof is complete. □

Remark 2.3. For $\alpha = 1$ in inequality (2.26), we obtain

$$a^2 + b^2 + c^2 \geq \frac{1}{2} \sum_{cyclic} \sqrt{F_\lambda (a^2, b^2)(n)} \geq 4\sqrt{3}\Delta, \tag{2.33}$$

with $\lambda \in [0, 1]$, which proves Weitzenböck's Inequality.

Corollary 2.4. *In any triangle ABC, there are the following inequalities:*

$$R - 2r \geq 3Rr \left| \sqrt[3]{E_\lambda \left(\frac{1}{h_a}, \frac{1}{h_b}, \frac{1}{h_c}\right)} - \sqrt[3]{E_{1-\lambda} \left(\frac{1}{h_a}, \frac{1}{h_b}, \frac{1}{h_c}\right)} \right| \geq 0, \tag{2.34}$$

$$s - 3\sqrt{3}r \geq \frac{3}{\sqrt{s}} \left| \sqrt{3E_\lambda (s-a, s-b, s-c)} - \sqrt{3E_{1-\lambda} (s-a, s-b, s-c)} \right| \geq 0, \tag{2.35}$$

$$a^{2\alpha} + b^{2\alpha} + c^{2\alpha} - 3\left(\frac{4\Delta}{\sqrt{3}}\right)^\alpha$$
$$\geq 3 \left| \sqrt[3]{E_\lambda (a^{2\alpha}, b^{2\alpha}, c^{2\alpha})} - \sqrt[3]{E_{1-\lambda} (a^{2\alpha}, b^{2\alpha}, c^{2\alpha})} \right| \geq 0, \tag{2.36}$$

$$R - 2r \geq \frac{Rr}{2} \sum_{cyclic} \left| \sqrt{F_\lambda \left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} - \sqrt{F_{1-\lambda} \left(\frac{1}{h_a}, \frac{1}{h_b}\right)(n)} \right| \geq 0, \tag{2.37}$$

$$s-3\sqrt{3}r \geq \frac{1}{2}\sum_{cyclic}\left|\sqrt{F_{\lambda}\left(s-a,s-b\right)(n)}-\sqrt{F_{1-\lambda}\left(s-a,s-b\right)(n)}\right| \geq 0 \tag{2.38}$$

and

$$a^{2\alpha}+b^{2\alpha}+c^{2\alpha}-3\left(\frac{4\Delta}{\sqrt{3}}\right)^{\alpha} \geq \frac{1}{2}\sum_{cyclic}\left|\sqrt{F_{\lambda}\left(a^{2\alpha},b^{2\alpha}\right)(n)}-\sqrt{F_{1-\lambda}\left(a^{2\alpha},b^{2\alpha}\right)(n)}\right| \geq 0 \tag{2.39}$$

for all integers $n \geq 0$, *for all* $\alpha > 0$ *and* $\lambda \in [0,1]$.

Proof. These all follow trivially from Corollary 2.1 and Corollary 2.3 taking into account that if $\lambda \in [0,1]$, then $1-\lambda \in [0,1]$. □

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REFERENCES

[1] Bencze, M., Minculete, N. and Pop, O. T., *New inequalities for the triangle*, Octogon Mathematical Magazine **17** (2009), No.1, 70-90
[2] Bottema, O., Djordjević, R. Z., Janic, R. R., Mitrinović, D. S. and Vasić, P. M., *Geometric Inequalities*, Gröningen, 1969
[3] Carlitz, L. and Leuenberger, F., Problem E 1454, *Amer Math. Monthly* **68** (1961), 177
[4] Djukić, D., Janković, V., Matić, I. and Petrović, N., *The IMO Compendium*, Springer, 2006
[5] Edwards, N. A., *Problem 1273*, Nouv. Ann. Math. **37** (1878), 475
[6] Euler, L., *Solutio facilis problematum quorundam geometricorum difficillimorum*, Novi Commentarii academiae scientiarum imperialis Petropolitanae **11**(1767), 103-123 (Reprinted in *Opera Omnia*, I 26, 139-157, 1953)
[7] Mills, C. N. and Dunkel, O., *Problem 3207*, Amer Math. Monthly **34** (1927), 382-384
[8] Mitrinović, D. S., *Analytic Inequalities*, Springer Verlag Berlin, Heidelberg, New York, 1970
[9] Padoa, A., *Una questione di minimo*, Period. Mat. **5** (1925), No. 4, 80-85
[10] Pólya, G. and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, II, Leipzig, 1925
[11] Sándor, J., *On the Geometry of Equilateral Triangles*, Forum Geometricorum **5** (2005), 107-117
[12] Weitzenböck, R., *Über eine Ungleichung in der Dreiecksgeometrie*, Mathematische Zeitschrift **5** (1919), No. 1-2, 137-146

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