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On the generalized Coşniţă-Turtoiu inequalities

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ABSTRACT.
The aim of this note is twofold: first, to give a simple proof to a simplified version of a result of Y.-D. Wu and M. Bencze [*The refinement and generalization of a double Coşniţă-Turtoiu inequality with one parameter*, Creat. Math. Inf. **19** (2010), No. 1, 96–100] and, secondly, to extend this one parameter Coşniţă-Turtoiu inequality to a two parameters inequality.

1. INTRODUCTION

For a given triangle ABC, let a, b, c denote the side-lengths, h_a, h_b, h_c the altitudes lengths, s the semiperimeter, Δ the area, R the circumradius and r the inradius, respectively. To simplify the writing of some expressions we will also use the cyclic sum notation in a triangle, that is, $\sum f(a) = f(a) + f(b) + f(c)$ etc.

In 1961, C. Coşniţă and F. Turtoiu proposed the inequality

$$6 \leq \sum \frac{h_a + r}{h_a - r}, \tag{1.1}$$

as Problem 33. 1) at page 158, in [3]. In the third edition of [3], that is, in the problems book [4], it appears as Problem 75 at page 162. This inequality has been completed in 2003 to a double inequality by Tian [6], who showed that

$$\sum \frac{h_a + r}{h_a - r} < 7. \tag{1.2}$$

Inequality (1.2) is actually a particular case of the right hand side of the following beautiful refined Coşniţă-Turtoiu double inequality, obtained in 1998 by Zhang [8]:

$$\frac{19}{3} - \frac{2r}{3R} \leq \sum \frac{h_a + r}{h_a - r} \leq 7 - \frac{2r}{R}. \tag{1.3}$$

On the other hand, Chu [2] generalized the original Coşniţă-Turtoiu inequality (1.1) by introducing a parameter λ and proved that for, any $\lambda \leq 2$, the following inequality holds:

$$\frac{3(3+\lambda)}{3-\lambda} \leq \sum \frac{h_a + \lambda r}{h_a - \lambda r}. \tag{1.4}$$

Clearly, (1.1) is obtained from (1.4) for $\lambda = 1$.

By combining Zhang's and Chu's inequalities, Wu and Bencze [7] proved very recently the following refined Coşniţă-Turtoiu double inequality with one parameter.

Theorem 1.1. *In any triangle ABC we have:*

$$\frac{6+\lambda}{2-\lambda} - \frac{4\lambda^2}{(\lambda-2)(\lambda-3)} \cdot \frac{r}{R} \geq \sum \frac{h_a + \lambda r}{h_a - \lambda r} \geq E(\lambda, R, r), \tag{1.5}$$

where

$$E(\lambda, R, r) = \frac{3(3+\lambda)}{3-\lambda} + \frac{2\lambda^2}{(4-\lambda)(3-\lambda)(2-\lambda)} \cdot \left(1 - \frac{2r}{R}\right), \text{ if } \lambda < \frac{7-\sqrt{17}}{2}$$

and

$$E(\lambda, R, r) = \frac{3(3+\lambda)}{3-\lambda} + \frac{16(\lambda-1)}{(3-\lambda)^3} \cdot \left(1 - \frac{2r}{R}\right), \text{ if } \frac{7-\sqrt{17}}{2} \leq \lambda \leq 2.$$

Remark 1.1. It is easy to check that for $\lambda = 1 < \dfrac{7-\sqrt{17}}{2}$, by the inequality (1.5) we obtain exactly the double inequality (1.3) and that, in view of Euler's inequality, $R \geq 2r$, (1.4) is also a consequence of (1.5).

The aim of this paper is to give a simpler proof than the one in [7] for a simplified version of the right hand side of (1.5), and simultaneously, to obtain a two parameters version of this inequality.

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2. The main result

Our main result is a simplified (and weaker) version of the right-hand side inequality (1.5) with two parameters.

Theorem 2.2. *In any triangle ABC, for any $\mu \le 2$ and $\lambda \in \mathbb{R}$ satisfying $(\lambda+\mu)\mu \ge 0$, we have*

$$\frac{3(3+\lambda)}{3-\mu} \le \sum \frac{h_a+\lambda r}{h_a-\mu r}. \tag{2.6}$$

Proof. By using the well known identity in a triangle $\Delta = rs$, we get $h_a = \dfrac{2\Delta}{a} = \dfrac{2rs}{a}$ as well as the other two similar ones, and hence

$$\sum \frac{h_a+\lambda r}{h_a-\mu r} = \sum \frac{\frac{2rs}{a}+\lambda r}{\frac{2rs}{a}-\mu r} = \sum \frac{(\lambda+1)a+b+c}{(1-\mu)a+b+c}. \tag{2.7}$$

Denote

$$(1-\mu)a+b+c = x,\ a+(1-\mu)b+c = y,\ a+b+(1-\mu)c = z. \tag{2.8}$$

From $\mu \le 2$, it follows that $x = (1-\mu)a+b+c \ge -a+b+c > 0$, by the fundamental triangle inequality. So, $x, y, z > 0$.

If $\mu = 0$, then $x = y = z$ and

$$\sum \frac{(\lambda+1)a+b+c}{(1-\mu)a+b+c} = \sum \frac{(\lambda+1)a+b+c}{a+b+c} = \lambda+3,$$

which by (2.7) shows that (2.6) is indeed true in this case.

Assume by now that $\mu \ne 0$. By summing the three identities in (2.8) we get

$$a+b+c = \frac{x+y+z}{3-\mu} \text{ which implies } a = \frac{(\mu-2)x+y+z}{\mu(3-\mu)}$$

and hence

$$(1+\lambda)a+b+c = \frac{x+y+z}{3-\mu} + \lambda\cdot\frac{(\mu-2)x+y+z}{\mu(3-\mu)} =$$
$$= \frac{(\lambda\mu+\mu-2\lambda)x+(\lambda+\mu)y+(\lambda+\mu)z}{\mu(3-\mu)}$$

and the other similar ones. Therefore

$$\sum \frac{(\lambda+1)a+b+c}{(1-\mu)a+b+c} = 3\cdot\frac{\lambda\mu+\mu-2\lambda}{\mu(3-\mu)} + \frac{\lambda+\mu}{\mu(3-\mu)}\cdot F(x,y,z) \tag{2.9}$$

where

$$F(x,y,z) = \left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{z}{y}+\frac{y}{z}\right) \ge 2\cdot 3, \tag{2.10}$$

since x, y, z are positive numbers. Using the fact that, by hypothesis, $\dfrac{\lambda+\mu}{\mu(3-\mu)} \ge 0$, by (2.7), (2.9) and (2.10) we now get the desired conclusion, i.e.,

$$\sum \frac{h_a+\lambda r}{h_a-\mu r} \ge 3\cdot\frac{\lambda\mu+\mu-2\lambda}{\mu(3-\mu)} + 6\cdot\frac{\lambda+\mu}{\mu(3-\mu)} = \frac{3(\lambda+3)}{3-\mu}.$$

□

Remark 2.2. If $\lambda = \mu \le 2$, then $\dfrac{\lambda+\mu}{\mu(3-\mu)} = \dfrac{2}{3-\mu} > 0$ and by inequality (2.6) we get exactly the inequality (1.4).

Note that in the same books [3] and [4], Coşniţă and Turtoiu also included the inequality

$$\sum \frac{h_a-r_a}{h_a+r_a} \le 0, \tag{2.11}$$

as Problem 33. 2), p. 158, in [3] and as Problem 76, p. 162, in [4], respectively. The corresponding result to inequality (2.11) similar to that in inequality (1.4) is given by

Theorem 2.3. *In any triangle ABC, for any $\lambda \ge 0$ we have*

$$\sum \frac{h_a-\lambda r_a}{h_a+\lambda r_a} \le \frac{3(1-\lambda)}{\lambda+1}\ (0 \le \lambda \le 2); \tag{2.12}$$

and

$$\sum \frac{h_a-\lambda r_a}{h_a+\lambda r_a} \ge \frac{3(1-\lambda)}{\lambda+1}\ (\lambda > 2). \tag{2.13}$$

Proof. We have

$$\frac{h_a - \lambda r_a}{h_a + \lambda r_a} = \frac{\frac{2\Delta}{a} - \lambda\frac{\Delta}{s-a}}{\frac{2\Delta}{a} + \lambda\frac{\Delta}{s-a}} = \frac{-(\lambda+1)a + b + c}{(\lambda-1)a + b + c}.$$

Denote

$$x = (\lambda-1)a + b + c;\ y = a + (\lambda-1)b + c;\ y = a + b + (\lambda-1)c.$$

From the assumption $\lambda \geq 0$, it follows that $x, y, z > 0$. Similarly to the proof of Theorem 2.2, we get

$$a + b + c = \frac{x+y+z}{\lambda+1};$$

and for $\lambda \neq 2$, we get

$$a = \frac{\lambda x - y - z}{(\lambda-2)(\lambda+1)}$$

and

$$-(\lambda+1)a + b + c = \frac{-(\lambda^2+\lambda+2)x + 2\lambda y + 2\lambda z}{(\lambda-2)(\lambda+1)}.$$

Therefore

$$\sum \frac{h_a - \lambda r_a}{h_a + \lambda r_a} = \frac{-3(\lambda^2+\lambda+2)}{(\lambda-2)(\lambda+1)} + \frac{2\lambda}{(\lambda-2)(\lambda+1)} \cdot F(x,y,z),$$

where $F(x,y,z)$ satisfies (2.10). If $\lambda < 2$ then

$$\frac{2\lambda}{(\lambda-2)(\lambda+1)} \cdot F(x,y,z) \leq \frac{12\lambda}{(\lambda-2)(\lambda+1)}$$

and hence

$$\sum \frac{h_a - \lambda r_a}{h_a + \lambda r_a} \leq \frac{3(1-\lambda)}{\lambda+1},$$

which is (2.12). Note that if $\lambda = 2$, then $x = y = z = a + b + c$, $\dfrac{3(1-\lambda)}{\lambda+1} = -1$ and

$$\sum \frac{h_a - \lambda r_a}{h_a + \lambda r_a} = \sum \frac{-3a + b + c}{a + b + c} = -1.$$

If $\lambda > 2$ then

$$\frac{2\lambda}{(\lambda-2)(\lambda+1)} \cdot F(x,y,z) \geq \frac{12\lambda}{(\lambda-2)(\lambda+1)}$$

and hence

$$\sum \frac{h_a - \lambda r_a}{h_a + \lambda r_a} \geq \frac{3(1-\lambda)}{\lambda+1},$$

which is (2.13). This completes the proof. □

Remark 2.3. 1) If $\lambda = 1$ then by Theorem 2.3 we obtain exactly the inequality (2.11);
2) Theorem 2.3 can be further extended to a two parameter inequality, similarly to Theorem 2.2.

Theorem 2.4. *In any triangle ABC, for $\mu \geq 0$ and $\lambda \in \mathbb{R}$ satisfying $\lambda + \mu \geq 0$, we have*

$$\sum \frac{h_a - \lambda r_a}{h_a + \mu r_a} \leq \frac{3(1-\lambda)}{\mu+1}\ (0 \leq \mu \leq 2); \tag{2.14}$$

and

$$\sum \frac{h_a - \lambda r_a}{h_a + \mu r_a} \geq \frac{3(1-\lambda)}{\mu+1}\ (\mu > 2). \tag{2.15}$$

Proof. Here if $\mu \neq 2$, we have

$$\sum \frac{h_a - \lambda r_a}{h_a + \mu r_a} = \frac{-3(\lambda\mu+\mu+2)}{(\mu-2)(\mu+1)} + \frac{\lambda+\mu}{(\mu-2)(\mu+1)} \cdot F(x,y,z),$$

where $F(x,y,z)$ satisfies (2.10), while for $\mu = 2$,

$$\sum \frac{h_a - \lambda r_a}{h_a + \mu r_a} = 1 - \lambda.$$

The rest of the proof is similar to that of Theorem 2.3 □

Remark 2.4. 1) If we take $\mu = \lambda$, by Theorem 2.4 we obtain Theorem 2.3;
2) The following problem also naturally arise: extend the two parameter inequality (2.6) in the same way in which the inequality (1.4) has been extended and refined to obtain (1.5).

Hint. Similarly to [7] we get

$$\sum \frac{h_a + \lambda r}{h_a - \mu r} =$$
$$= \frac{[(\lambda+4)(\lambda-2\mu)+12]s^2 + 2\mu(2\mu+3\lambda\mu-4\lambda)Rr + \mu(\mu-2\lambda)r^2}{(\mu-2)^2 s^2 + 2\mu^2(2-\mu)Rr + \mu^2 r^2}.$$

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