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Empirical study of a Padé type accelerating method of Picard iteration

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ABSTRACT.
We use a Padé type acceleration technique for the method of successive approximations in [J. Biazar and A. Amirteimoori, *An improvement to the fixed point iterative method*, Applied Mathematics and Computation **182** (2006), 567-571, doi:10.1016/j.amc.2006.04.019] to empirically study the possibility of accelerating Picard iteration for some other known test functions.

1. INTRODUCTION

Recently, Biazar and Amirteimoori considered in [9] a Padé-type technique to accelerate Picard iteration method for solving three scalar equations of the form

$$f(x) = 0 \tag{1.1}$$

which were equivalently written as a fixed point problem

$$x = g(x), \tag{1.2}$$

where $g : [a, b] \to [a, b]$ is the iteration function.

Under appropriate assumptions on f (and therefore on g), the Picard iteration (or the sequence of successive approximations, as it is generally known), i.e.,

$$x_{n+1} = g(x_n), \quad n \geq 0, \tag{1.3}$$

converges to the (unique) fixed point of g, say α, which is the (unique) solution of (1.1) in the interval $[a, b]$.

Note that for a certain nonlinear equation (1.1), the fixed point problem (1.2) is not uniquely defined. For example, the equation $x^3 + 4x^2 - 10 = 0$ can be written under a fixed point form as $x = \frac{1}{2}\sqrt{10 - x^3}$ or $x = \sqrt[3]{10 - 4x^2}$.

As the convergence order of the Picard iteration (1.3) is generally linear (see for example Berinde [6]), the method converges rather slowly to the fixed point α.

In order to improve the convergence speed of (1.3), the authors in [9] considered the following equivalent fixed point problem

$$x = g_\lambda(x) \tag{1.4}$$

with g_λ of the form

$$g_\lambda(x) = \frac{g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1}}, \tag{1.5}$$

where $k \in \mathbb{N},\ k \geq 2$ and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k \in \mathbb{R}$ are parameters that should be determined in such a way that the new iteration function g_λ will yield a faster Picard iteration.

Note that the method of constructing (1.5) is rather similar to the way in which the Padé approximant of order (m, n), $[m/n]_f(x)$, is obtained, see for example [5]:

$$[m/n]_f(x) = \frac{p_0 + p_1 x + p_2 x^2 + \ldots + p_m x^m}{1 + q_1 x + q_2 x^2 + \ldots + q_n x^n}. \tag{1.6}$$

This is the reason we shall name in the following (1.5) as a Padé type transform.

The aim of this paper is twofold: first, to derive the convergence order of the Picard iteration associated to (1.4) and secondly, to perform a similar empirical study of the rate of convergence for other values of k, in the case of the equations from [9], as well as for other test functions taken from literature. This will allow us to infer which value of k is optimal for each equation.

2. THE PADÉ-TYPE ACCELERATION OF THE PICARD ITERATION

This result is taken from [9].
Based on the fact that the fixed point equation

$$x = g(x)$$

is equivalent to

$$x + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k = g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k,$$

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which can be written under the form

$$x = g_\lambda(x) = \frac{g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k}{1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1}} \tag{2.7}$$

we get exactly the fixed point problem (1.4).
It is tacitely assumed that $g_\lambda(x)$ is well defined on the interval $[a, b]$ where the original equation is solved, that is, the equation

$$1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1} = 0$$

has no real roots on $[a, b]$.
The main idea of constructing such an accelerated method is to determine the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that the new iteration function g_λ satisfies

$$g_\lambda^{(i)}(\alpha) = 0, \;\; i = 1, 2, \ldots, k, \tag{2.8}$$

where α is the unique solution of (1.1) and (1.2) in the interval $[a, b]$.

Using (2.7), the equation (2.8) yields an upper diagonal linear system of equations with the unknowns $\lambda_1, \lambda_2, \ldots, \lambda_k$ which always is uniquely solvable, as in the case of the original Padé transform. Indeed, by (2.7) we have

$$g_\lambda(x)(1 + \lambda_1 + \lambda_2 x + \lambda_3 x^2 + \ldots + \lambda_k x^{k-1}) = g(x) + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 + \ldots + \lambda_k x^k$$

which, by differentiating with respect to x, gives

$$g'_\lambda(x)(1 + \lambda_1 + \lambda_2 x + \ldots + \lambda_k x^{k-1}) + g_\lambda(x)(\lambda_2 + 2\lambda_3 x + \ldots + (k-1)\lambda_k x^{k-2}) =$$

$$= g'(x) + \lambda_1 + 2\lambda_2 x + \ldots + k\lambda_k x^{k-1}. \tag{2.9}$$

If we take $x = \alpha$ in (2.9) and use the fact that $g_\lambda(\alpha) = g_\lambda(x) = \alpha$ and $g'_\lambda(\alpha)$ is required to be zero, we get the linear equation

$$\lambda_1 + 2\lambda_2 \alpha + \ldots + k\lambda_k \alpha^{k-1} = -g'(\alpha).$$

Now we differentiate again (2.9) and then, by letting $x = \alpha$, we are lead to the linear equation

$$2\lambda_2 + 3\lambda_3 \alpha + \ldots + k(k-1)\lambda_k \alpha^{k-1} = -g''(\alpha)$$

and so on. The generic formula for the i^{th} derivative of g_λ is

$$-g^{(j)}(\alpha) = \sum_{i=j}^{k} i(i-1)(i-2)\ldots(i-j+1)\lambda_i \alpha^{i-j}, \;\; j = 1, 2, \ldots, k. \tag{2.10}$$

If we rewrite the linear $k \times k$ system (2.10) in a matrix form we have

$$\begin{pmatrix} 1 & 2\alpha & 3\alpha^2 & \ldots & k\alpha^{k-1} \\ 0 & 2 & 6\alpha & \ldots & k(k-1)\alpha^{k-2} \\ 0 & 0 & 6 & \ldots & k(k-1)(k-2)\alpha^{k-3} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & k! \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_k \end{pmatrix} = \begin{pmatrix} -g'(\alpha) \\ -g^{(2)}(\alpha) \\ -g^{(3)}(\alpha) \\ \vdots \\ -g^{(k)}(\alpha) \end{pmatrix}. \tag{2.11}$$

By solving (2.11), we can uniquely find the values of $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ and hence get the iteration function of the accelerated process

$$x_{n+1} = g_\lambda(x_n), \; n \geq 0.$$

We end this section by reminding the concept of convergence order that will be used in the paper.

Let $\{x_n\} \subset \mathbb{R}$ be a sequence of real numbers convergent to $\alpha \in \mathbb{R}$ (which is obtained by iterating a fixed point equation)

Definition 2.1. [13]Let $\{x_n\}$ converge to α. If there exist an integer constant p, and a real positive constant C such that

$$\lim_{n\to\infty} \left| \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} \right| = C,$$

then p is called the order and C the constant of convergence.

The concept of rate of convergence given by Definition 2.1 is also known as the Q-order of convergence, see the monographs by Măruşter [13] and Ortega and Rheinboldt [14].

The next theorem shows how the fixed point iteration defined by the function g_λ accelerates the fixed point iteration defined by g.

Theorem 2.1. *Let $g \in C^{k+1}[a, b]$ such that the associated iteration function g_λ satisfy* (2.8)*, where α is the unique solution in $[a, b]$ of* (1.2)*. Then the accelerated Picard iteration*

$$x_{n+1}^\lambda = g(x_n^\lambda), \;\; n \geq 0$$

has Q-order of convergence k.

Proof. By the Taylor expansion of g_λ at x we find

$$g_\lambda(x_n) = g_\lambda(x) + \frac{g'_\lambda(x)}{1!}(x_n - x) + \ldots + \frac{g_\lambda^{(k)}(x)}{k!}(x_n - x)^k + \ldots$$

which yields, in view of $g_\lambda(\alpha) = \alpha$ and (2.8)

$$g_\lambda(x_n) - \alpha = \frac{g_\lambda^{(k+1)}(x)}{(k+1)!}(x_n - \alpha)^{k+1} + \ldots$$

that is

$$\lim_{n\to\infty} \frac{|x_{n+1} - \alpha|}{|x_n - \alpha|^{k+1}} = \frac{g_\lambda^{(k+1)}(\alpha)}{(k+1)!},$$

which completes the proof. □

Remark 2.1. Note that, generally, $g'(\alpha) \neq 0$, so (x_n) has the Q-order of convergence equal to 1, see the Examples in the next section.

3. Some useful fixed point theorems

In this section we present three known results in fixed point theory, taken from [6], that ensure, under various assumptions, the existence and uniqueness of a fixed point of a mapping g as well as the convergence of the Picard iteration to that fixed point. For two of them, the rate of convergence is also given.

Theorem 3.2 (Contraction Mapping Principle). *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \text{for all } x, y \in X\,, \tag{3.12}$$

where $0 \le a < 1$ is constant. Then:

$(p1)$ T has a unique fixed point x^* in X;

$(p2)$ The Picard iteration $\{x_n\}_{n=0}^\infty$ defined by

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \ldots \tag{3.13}$$

converges to x^*, for any $x_0 \in X$.

$(p3)$ The following estimate holds:

$$d(x_{n+i-1}, x^*) \le \frac{a^i}{1-a}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots \tag{3.14}$$

$(p4)$ The rate of convergence of Picard iteration is given by

$$d(x_n, x^*) \le a\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots \tag{3.15}$$

Theorem 3.3 (Zamfirescu's Mapping Principle). *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping for which there exist $a \in [0, 1)$, $b, c \in [0, \frac{1}{2})$ such that for all $x, y \in X$, at least one of the following conditions is true:*

(z_1) $d(Tx, Ty) \le a\, d(x, y)$;

(z_2) $d(Tx, Ty) \le b\big[d(x, Tx) + d(y, Ty)\big]$;

(z_3) $d(Tx, Ty) \le c\big[d(x, Ty) + d(y, Tx)\big]$.

Then the Picard iteration $\{x_n\}$ defined by (3.13) and starting from $x_0 \in X$ converges to the unique fixed point x^* of T with the following error estimate

$$d(x_{n+i-1}, x^*) \le \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots$$

where $\delta = \max\left\{a, \dfrac{b}{1-b}, \dfrac{c}{1-c}\right\}$.

Moreover, the convergence rate of the Picard iteration is given by

$$d(x_n, x^*) \le \delta \cdot d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots \tag{3.16}$$

Theorem 3.4 (Almost Contraction Mapping Principle). *Let (X, d) be a complete metric space and $T : X \to X$ an almost contraction, that is, a mapping for which there exist a constant $\delta \in (0, 1)$ and some $L \ge 0$ such that*

$$d(Tx, Ty) \le \delta \cdot d(x, y) + L d(y, Tx)\,, \quad \text{for all } \; x, y \in X\,. \tag{3.17}$$

Then

1) $F(T) = \{x \in X : Tx = x\} \neq \emptyset$;

2) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^\infty$ given by (1.2) converges to some $x^* \in F(T)$;

3) The following estimate holds

$$d(x_{n+i-1}, x^*) \le \frac{\delta^i}{1-\delta}\, d(x_n, x_{n-1})\,, \quad n = 0, 1, 2, \ldots; \; i = 1, 2, \ldots \tag{3.18}$$

4. Numerical examples

Example 4.1. [9]Test function: $f(x) = x^3 + 4x^2 - 10$, which has a unique root in the interval (1,2). We use an approximate value for α, $\alpha \cong 1.5$ and $g(x) = \frac{1}{2}\sqrt{10 - x^3}$. The values of the parameters λ_i involved in (2.7) are
For $k = 2$:

$$\lambda_1 = -1.15660903,\ \lambda_2 = 1.20815133.$$

For $k = 3$:

$$\lambda_1 = 1.57623135,\ \lambda_2 = -2.43563586,\ \lambda_3 = 1.21459573.$$

For $k = 4$:

$$\lambda_1 = -3.122090855,\ \lambda_2 = 6.961008590,\ \lambda_3 = -5.049833910,\ \lambda_4 = 1.392095477.$$

and for $k = 5$:

$$\lambda_1 = 6.176012965,\ \lambda_2 = -17.83393493,\ \lambda_3 = 19.74510962,\ \lambda_4 = -9.627879427,$$

$\lambda_5 = 1.836662484$. The results for the three fastest methods used in Example 4.1 are listed in Table 1.

Table 1

n	$k = 2$	$k = 3$	$k = 4$	$k = 5$	
	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g(x_n)$
0	1.5	1.5	1.5	1.5	1.5
1	1.37131921	1.37131920	1.37131921	1.37131923	1.28695377
2	1.36517040	1.36525174	1.36523987	1.36524189	1.40254080
3	1.36523078	1.36523005	1.36523000	1.36523001	1.34545838
4	1.36523000	1.36523001	1.36523001		1.37517025
5	1.36523001				1.36009419
6					1.36784697
$\vdots$					$\vdots$
25					1.36523001

For Example 4.1 we observe that for $k = 5$ we have the best rate of convergence.

Example 4.2. [9]Test function $f(x) = x - \tan x = 0$. This equation has a root which lies near $\dfrac{3\pi}{2}$. Let $g(x) = \tan x$, then $g'(x) = 1 + \tan^2 x \geq 1$, which is not a suitable $g(x)$. Let $\alpha \cong 4.5$ and $g(x) = \tan x$. We show that the new technique works even in this case. The values of the parameters λ_i involved in (2.7) are
For $k = 3$:

$$\lambda_1 = -28939.740120,\ \lambda_2 = 13060.829480,\ \lambda_3 = -1474.394932.$$

For $k = 4$:

$$\lambda_1 = 814467.2540,\ \lambda_2 = -54910.4993,\ \lambda_3 = 123474.7892,\ \lambda_4 = -9255.495122.$$

For $k = 5$:

$$\lambda_1 = -2.152270898 \cdot 10^7,\ \lambda_2 = 1.930605732 \cdot 10^7,\ \lambda_3 = -6.494947820 \cdot 10^6,$$

$\lambda_4 = 9.712515583 \cdot 10^5,\ \lambda_5 = -54472.61408.$
and for $k = 6$:

$$\lambda_1 = 5.464009450 \cdot 10^8,\ \lambda_2 = -6.117202249 \cdot 10^8,\ \lambda_3 = 2.739611774 \cdot 10^8,$$

$\lambda_4 = -6.135233180 \cdot 10^7,\ \lambda_5 = 6.870369979 \cdot 10^6,\ \lambda_6 = -3.077707818 \cdot 10^5.$

The results for the three fastest methods used in Example 4.2 are listed in Table 2.
Table 2

n	$k = 3$	$k = 4$	$k = 5$	$k = 6$	
	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g(x_n)$
0	4.5	4.5	4.5	4.5	4.5
1	4.493616	4.493280632	4.488372093	5.444444444	4.637332
2	4.493410	4.493716711	4.487779511	5.444305527	13.298192
3		4.493170168	4.479895561	5.444414683	0.898203
4		4.493888939	4.487352445	5.444477321	1.255520
5		4.493311705	4.495007882	5.444416274	
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
10					-0.076296
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
25		4.493647770	4.504339881		

For Example 4.2 we observe that for $k = 3$ we have the best rate of convergence.

Example 4.3. [9]

Test function $f(x) = x - 3^{-x} = 0$. $f(x)$ is continuous on $[\frac{1}{3}, 1]$ and $f(\frac{1}{3}) \cdot f(1) < 0$. By Weierstrass' theorem, α, the root of $f(x)$, lies in $(\frac{1}{3}, 1)$. Let $\alpha \cong 0.6 \in (\frac{1}{3}, 1)$ and $g(x) = 3^{-x}$. The values of the parameters λ_i involved in (2.7) are
For $k = 5$:

$$\lambda_1 = 1.0979516,\ \lambda_2 = -1.2013119,\ \lambda_3 = 0.6435174,\ \lambda_4 = -0.2083743,\ \lambda_5 = 0.0344936.$$

For $k = 6$:

$$\lambda_1 = 1.0985408,\ \lambda_2 = -1.2062230,\ \lambda_3 = 0.6598879,\ \lambda_4 = -0.2356586,\ \lambda_5 = 0.0572306,$$

$\lambda_6 = -0.00755790$.
For $k = 7$:

$$\lambda_1 = 1.0986056,\ \lambda_2 = -1.2068705,\ \lambda_3 = 0.6625857,\ \lambda_4 = -0.2416536,\ \lambda_5 = 0.0647243,$$

$\lambda_6 = -0.0125748,\ \lambda_7 = 0.0013877$.
and for $k = 8$:

$$\lambda_1 = 1.0986117,\ \lambda_2 = -1.2069416,\ \lambda_3 = 0.6629413,\ \lambda_4 = -0.2426415,\ \lambda_5 = 0.0663709,$$

$\lambda_6 = -0.0142213,\ \lambda_7 = 0.0023024,\ \lambda_8 = 0.0002177$.

The results for the three fastest methods used in Example 4.3 are listed in Table 3.
Table 3

n	$k = 5$	$k = 6$	$k = 7$	$k = 8$	
	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g_\lambda(x_n)$	$x_{n+1} = g(x_n)$
0	0.33333	0.33333	0.33333	0.33333	0.33333
1	0.53769	0.5376929282	0.5376928610	0.5376928590	0.69336
2	0.54779	0.5477874354	0.5477874346	0.5477874349	0.46686
3	0.54781	0.5478086216	0.5478086216	0.5478086223	0.59876
4	0.54781	0.5478086219	0.5478086213	0.5478086215	0.51799
5		0.5478086217	0.5478086215	0.5478086219	
6			0.5478086213	0.5478086214	
7			0.5478086215	0.5478086216	
8				0.5478086218	
9				0.5478086218	
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
21					0.54781

For the Example 4.3 we observe that for $k = 5$ we have the best rate of convergence.

Example 4.4. [11]Test function: $f(x) = (x-1)^3 - 1 = 0$. We observe that $x = 2$ is a root of $f(x)$. We use an approximative value for α, $\alpha \cong 1.7$ and $g(x) = \sqrt[3]{3x^2 - 3x + 2}$. Note that g is a contraction on $\mathbb{R}$. The values of the parameters λ_i involved in (2.7) are
For $k = 2$:

$$\lambda_1 = -0.8007005397,\ \lambda_2 = 0.0217115888.$$

For $k = 3$:

$$\lambda_1 = -0.4719973830,\ \lambda_2 = -0.3649980073,\ \lambda_3 = 0.1137381165.$$

For $k = 4$:

$$\lambda_1 = 0.2083667801,\ \lambda_2 = -1.565640648,\ \lambda_3 = 0.8199984934,\ \lambda_4 = -0.1384824268.$$

For $k = 5$:

$$\lambda_1 = 1.180921712,\ \lambda_2 = -3.854005194,\ \lambda_3 = 2.839143681,\ \lambda_4 = -0.9303040692,$$

$\lambda_5 = 0.1164443592$.
and for $k = 6$:

$$\lambda_1 = 2.293100612,\ \lambda_2 = -7.125119605,\ \lambda_3 = 6.687513576,\ \lambda_4 = -3.194051066,$$

$\lambda_5 = 0.7822522995,\ \lambda_6 = -0.07833034592$.

The results for the five fastest methods used in the Example 4.4 are listed in Table 4.
Table 4
For the Example 4.4 we observe that for $k = 5$ we have the best rate of convergence.

Example 4.5. [11]Test function $f(x) = \cos x - x = 0$, which has a unique root in the interval $(0, 1)$. We use an approximative value for α, $\alpha \cong 0.5$ and $g(x) = \cos x$. Note that g is a contraction on $[0, 1]$. The values of the parameters λ_i involved in (2.7) are
For $k = 2$:

$$\lambda_1 = 0.04063425765,\ \lambda_2 = 0.8775825619.$$

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g(x_n)$
0	1.7	1.7	1.7	1.7	1.7	1.7
1	2.007486966	2.007486965	2.007486965	2.007486949	2.007487047	1.772631238
2	1.999773454	2.000100489	1.999981347	2.000012547	2.000006093	1.828035437
3	2.000006791	2.000001178	2.000000064	2.000000040	1.999999912	1.870174554
4	1.999999796	2.000000017	1.999999996	1.999999980	2.000000124	1.902133792
5	2.000000006	1.999999998	2.000000008	2.000000000	1.999999912	1.926313267
6	1.999999999	2.000000004	2.000000004	2.000000000	2.000000124	1.944570353
7	1.999999997	2.000000000	1.999999996			1.958333871
8	1.999999997		2.000000008			1.968697038
9			2.000000004			1.976492529
10			1.999999996			1.982352284
11			2.000000008			1.986754546
$\vdots$	$\vdots$		2.000000008	$\vdots$	$\vdots$	$\vdots$
32						1.999976320

For $k=3$:

$$\lambda_1 = -0.01929393468,\ \lambda_2 = 1.117295331,\ \lambda_3 = -0.2397127693.$$

For $k=4$:

$$\lambda_1 = -0.001010964635,\ \lambda_2 = 1.007597511,\ \lambda_3 = -0.02031712882,\ \lambda_4 = -0.1462637603.$$

For $k=5$:

$$\lambda_1 = 0.0002375393714,\ \lambda_2 = 0.9976094789,\ \lambda_3 = 0.00964697338,\ \lambda_4 = -0.1862158885,$$

$\lambda_5 = 0.01997606411.$
and for $k=6$:

$$\lambda_1 = 0.0000090022458559,\ \lambda_2 = 0.9998948502,\ \lambda_3 = 0.0005054823177,$$

$$\lambda_4 = -0.1679329185,\ \lambda_5 = 0.001693094069,\ \lambda_6 = 0.007313188016.$$

The results for the five fastest methods used in the Example 4.5 are listed in Table 5.
Table 5

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g(x_n)$
0	0.5	0.5	0.5	0.5	0.5	0.5
1	0.7552224168	0.7552224168	0.7552224168	0.7552224173	0.7552224180	0.8775825619
2	0.7393111553	0.7391639529	0.7391407837	0.73914159935	0.7391416688	0.6390124942
3	0.7390872396	0.739085247	0.7390851314	0.7390851334	0.7390851349	0.8026851007
4	0.7390851525	0.7390851333	0.7390851332	0.7390851335	0.7390851334	0.6947780268
5	0.7390851331	0.7390851327	0.7390851332	0.7390851341	0.7390851334	0.7681958313
6		0.7390851332		0.7390851335		0.719165449
7		0.7390851332		0.7390851341		
$\vdots$				$\vdots$		$\vdots$
52						0.7390852281

For the Example 4.5 we observe that for $k=2$ we have the best rate of convergence.

Example 4.6. [11]Test function $f(x) = (\sin x)^2 - x^2 + 1 = 0$. $f(x)$ is continuous on $[1,2]$ and $f(1) \cdot f(2) < 0$. By Weierstrass theorem, α, the root of $f(x)$, lies in $(1,2)$. Let $g(x) = \sqrt{1 + (\sin x)^2}$ and $\alpha \cong 1.5$. The values of the parameters λ_i involved in (2.7) are
For $k=2$:

$$\lambda_1 = -1.103967914,\ \lambda_2 = 0.7026746168.$$

For $k=3$:

$$\lambda_1 = -0.9630432881,\ \lambda_2 = 0.5147751162,\ \lambda_3 = 0.06263316685.$$

For $k=4$:

$$\lambda_1 = 0.03406382402,\ \lambda_2 = -1.479439108,\ \lambda_3 = 1.392109316,\ \lambda_4 = -0.2954391443.$$

For $k=5$:

$$\lambda_1 = -0.4994602957,\ \lambda_2 = -0.4317081221,\ \lambda_3 = 0.7193783305,\ \lambda_4 = -0.1631142617,$$

$\lambda_5 = 0.005723630675.$
and for $k=6$ we have the solutions

$$\lambda_1 = -0.3772313996,\ \lambda_2 = -0.8391377758,\ \lambda_3 = 1.262617869,\ \lambda_4 = -0.5252739538,$$

$\lambda_5 = 0.1264435280,\ \lambda_6 = -0.01609598632.$

The results for the five fastest methods used in the Example 4.6 are listed in Table 6.

Table 6

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g(x_n)$
0	1.5	1.5	1.5	1.5	1.5	1.5
1	1.407839387	1.407839386	1.407839387	1.407839385	1.407839388	1.412443361
2	1.404493085	1.404495094	1.404495963	1.404495477	1.404495477	1.405394334
3	1.404491648	1.404491648	1.404491651	1.404491648	1.404491650	1.404496296
4	1.404491648		1.404491647		1.404491648	1.404493062
5			1.404491649			1.404491813
6			1.404491646			
			1.404491646			
$\vdots$			$\vdots$			$\vdots$
9						1.404491648

For the Example 4.6 we observe that for $k = 2, 3, 5$ we have the best rate of convergence.

Example 4.7. [11] *Test function* $f(x) = e^{x^2+7x-30} - 1 = 0$*. We observe that* $x = 3$ *is a root for* $f(x)$*. Let* $g(x) = \sqrt{30 - 7x}$ *and an approximative value of* α, $\alpha \cong 2.5$*. The values of the parameters* λ_i *involved in* (2.7) *are*
For $k = 2$*:*

$$\lambda_1 = 0.2969848483,\ \lambda_2 = 0.2771858582.$$

For $k = 3$*:*

$$\lambda_1 = 1.024597726,\ \lambda_2 = -0.3049044440,\ \lambda_3 = 0.1164180604.$$

For $k = 4$*:*

$$\lambda_1 = 0.1757160352,\ \lambda_2 = 0.7137535850,\ \lambda_3 = -0.2910451512,\ \lambda_4 = 0.05432842822.$$

For $k = 5$*:*

$$\lambda_1 = 1.215596106,\ \lambda_2 = -0.9500545285,\ \lambda_3 = 0.7072397170,\ \lambda_4 = -0.2118808700,$$

$\lambda_5 = 0.02662092982.$
and for $k = 6$*:*

$$\lambda_1 = -0.09465278297,\ \lambda_2 = 1.670443250,\ \lambda_3 = -1.389158506,\ \lambda_4 = 0.6266784191,$$

$\lambda_5 = -0.1410909280,\ \lambda_6 = 0.01341694862.$

The results for the five fastest methods used in the Example 4.6 are listed in Table 7.

Table 7

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	
	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g_\lambda(x_n)$	$x_{n+1}=g(x_n)$
0	2.5	2.5	2.5	2.5	2.5	2.5
1	3.020382004	3.020382004	3.020382005	3.020382007	3.020382003	3.535533906
2	2.999645349	2.999947206	3.00009189	3.000037463	3.000042237	2.291563366
3	3.000006356	3.000000021	2.999999980	2.999999986	2.999999988	3.736182067
4	2.999999888	2.999999999	2.999999989	2.999999997	2.999999973	1.961307097
5	3.000000002	3.000000000	3.000000002	3.000000001	3.000000005	4.033714209
6	3.000000001		3.000000001	3.000000005	2.999999993	1.328156821
7	3.000000000		2.999999999	3.000000005	2.999999989	4.550044203
8			3.000000002		2.999999994	.
9			3.000000001		2.999999982	.
10			3.000000001		2.999999991	.
11					2.999999986	
12					3.000000007	
13					3.000000007	

For Example 4.7 we observe that for $k = 3$ we have the best rate of convergence.

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