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A weighted inequality involving the sides of a triangle

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ABSTRACT.

A new weighted inequality involving the sides of a triangle is proved by applying the method of the Difference Substitution, and, as an application, a geometric inequality involving an interior point of the triangle is derived. Finally, two related conjectures which are checked by computer are put forward.

1. INTRODUCTION

The earliest weighted inequality concerned with sides of a triangle appeared in J. Wolstenholme's book [1], and it is stated as follows: Let ABC be a triangle with sides a, b, c and let x, y, z be arbitrary real numbers. Then

$$(x-y)(x-z)a^2 + (y-z)(y-x)b^2 + (z-x)(z-y)c^2 \geq 0, \tag{1.1}$$

with inequality if and only if $x = y = z$.

Wolstenholme inequality (1.1) is one of the most important inequalities for the triangle, it has various equivalent versions and generalizations. The following important weighted inequality is actually its corollary:

$$(xa + yb + zc)^2 \geq (2bc + 2ca + 2ab - a^2 - b^2 - c^2)(yz + zx + xy). \tag{1.2}$$

with equality if and only if $x : y : z = (b + c - a) : (c + a - b) : (a + b - c)$.

Besides inequality (1.1) and (1.2), in literature (e.g [2], [3]) we see a few weighted inequalities involving the sides of a triangle. In 1992, the author established the following inequality with positive weights (see [4], [5]):

$$\frac{s-a}{x} + \frac{s-b}{y} + \frac{s-c}{z} \geq \frac{(xa + yb + zc)s}{yza + zxb + xyc}, \tag{1.3}$$

where $s = (a + b + c)/2$ and x, y, z are arbitrary positive real numbers. Equality holds in (1.3) if and only if $x = y = z$.

Later, the author [6] obtained again results involving acute triangles and arbitrary real numbers

$$(xa + yb + zc)^2 \geq 6\left[a(s-a)yz + b(s-b)zx + c(s-c)xy\right], \tag{1.4}$$

with equality if and only if the acute triangle ABC is equilateral and $x = y = z$.

The aim of this paper is to establish a new weighted inequality for arbitrary triangles, and then apply it and some simple well known results to deduce a new geometric inequality involving an interior point of a triangle. Finally, we propose two related conjectures.

Theorem 1.1. *Let ABC be a triangle with sides a, b, c and let x, y, z be non-negative real numbers among which at least one is not zero. Then*

$$\frac{y+z}{a(s-a)} + \frac{z+x}{b(s-b)} + \frac{x+y}{c(s-c)} \geq \frac{12(yz + zx + xy)}{xa^2 + yb^2 + zc^2}, \tag{1.5}$$

where $s = (a + b + c)/2$. Equality holds if and only if triangle ABC is equilateral and $x = y = z$.

2. THE PROOF OF THE THEOREM

It is easy to see that the inequality (1.5) may be transformed into to the following ternary quadratic inequality:

$$p_1x^2 + p_2y^2 + p_3z^2 \geq yzq_1 + zxq_2 + xyq_3, \tag{2.6}$$

and we have already known that inequality (2.6) holds for arbitrary real numbers x, y, z if and only if (e.g see [8], [7]): $p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, 4p_2p_3 - q_1 \geq 0, 4p_3p_1 - q_2^2 \geq 0, 4p_1p_2 - q_3^2 \geq 0$, and

$$4p_1p_2p_3 - (p_1q_1^2 + p_2q_2^2 + p_3q_3^2 + q_1q_2q_3) \geq 0. \tag{2.7}$$

However, inequality (1.5) actually holds for non-negative real numbers x, y, z only, but it may not be valid for some real numbers x, y, z. So, we can not apply the conditions which endures that (2.6) holds for arbitrary real numbers x, y, z to prove (1.5). On the other hand, at the moment we don't know what are the conditions of inequality (2.6) holding for all positive real numbers x, y, z. Therefore, it is not easy to prove our theorem. We find out that our theorem can be proved by employing the method of the Difference Substitution (see [10], [9]).

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Proof. We set $s-a=u, s-b=v, s-c=w$, then $a=v+w, b=w+u, c=u+v, u>0, v>0, w>0$, and the inequality (1.5) is equivalent to the following:

$$\frac{y+z}{u(v+w)}+\frac{z+x}{v(w+u)}+\frac{x+y}{w(u+v)}-\frac{12(yz+zx+xy)}{x(v+w)^2+y(w+u)^2+z(u+v)^2}\geq 0, \tag{2.8}$$

which involves three non-negative numbers x, y, z and three strictly positive numbers u, v, w. After some computations we found it is equivalent to

$$\frac{m_1x^2+m_2y^2+m_3z^2+n_1yz+n_2zx+n_3xy}{uvw(v+w)(w+u)(u+v)\left[x(v+w)^2+y(w+u)^2+z(u+v)^2\right]}\geq 0, \tag{2.9}$$

where

$$\begin{aligned}
m_1 &= u(uv+2vw+wu)(v+w)^3,\\
m_2 &= v(vw+2wu+uv)(w+u)^3,\\
m_3 &= w(wu+2uv+vw)(u+v)^3,\\
n_1 &= (w+u)(u+v)\left[u^2(v^2+4vw+w^2)+vw(v^2+w^2)\right.\\
&\quad \left.+u(v+w)(v^2-10vw+w^2)\right],\\
n_2 &= (u+v)(v+w)\left[v^2(w^2+4wu+u^2)+wu(w^2+u^2)\right.\\
&\quad \left.+v(w+u)(w^2-10wu+u^2)\right],\\
n_3 &= (v+w)(w+u)\left[w^2(u^2+4uv+v^2)\right.\\
&\quad \left.+uv(u^2+v^2)+w(u+v)(u^2-10uv+v^2)\right].
\end{aligned}$$

Clearly, to prove inequality (2.8) we need to prove that

$$m_1x^2+m_2y^2+m_3z^2+n_1yz+n_2zx+n_3xy\geq 0, \tag{2.10}$$

and hence by replacing $x\to x/(v+w), y\to y/(w+u), z\to z/(u+v)$, it suffices to show that

$$Q\equiv r_1x^2+r_2y^2+r_3z^2+s_1yz+s_2zx+s_3xy\geq 0, \tag{2.11}$$

where

$$\begin{aligned}
r_1 &= u(v+w)(uv+2vw+wu),\\
r_2 &= v(w+u)(vw+2wu+uv),\\
r_3 &= w(u+v)(wu+2uv+vw),\\
s_1 &= u^2(v^2+4vw+w^2)+vw(v^2+w^2)+u(v+w)(v^2-10vw+w^2),\\
s_2 &= v^2(w^2+4wu+u^2)+wu(w^2+u^2)+v(w+u)(w^2-10wu+u^2),\\
s_3 &= w^2(u^2+4uv+v^2)+uv(u^2+v^2)+w(u+v)(u^2-10uv+v^2).
\end{aligned}$$

In the sequel we shall apply the method of the Difference Substitution to prove inequality $Q\geq 0$.

Because of the symmetry in (1.5), without any loss of generality, suppose that $a\leq b\leq c$. This imply $u\geq v\geq w$, so we put

$$\begin{cases} v=w+p, & (p\geq 0)\\ u=w+p+q & (q\geq 0).\end{cases} \tag{2.12}$$

Under these hypotheses, we shall divide our argument into the following six cases:

Case 1. The non-negative real numbers x, y, z satisfy $x\geq y\geq z$.

In this case, we set

$$\begin{cases} y=z+m, & (m\geq 0)\\ x=z+m+n & (n\geq 0).\end{cases} \tag{2.13}$$

By putting (2.12) and (2.13) into the expression of Q, after analyzing we obtain the following identity:

$$k_1Q\equiv a_1w^4+a_2w^3+a_3w^2+a_4w+a_5+w^2Q_1^2, \tag{2.14}$$

where

$$\begin{aligned}
k_1 &= 12(p^2+pq+q^2),\\
a_1 &= (92m^2+92mn+95n^2)p^2+2q(46m^2+43mn+46n^2)p\\
&\quad +q^2(95m^2+92mn+92n^2),\\
a_2 &= (208m^2+208mn+268n^2)p^3+6q(52m^2+56mn+69n^2)p^2\\
&\quad +12q^2(27m^2+29mn+34n^2)p+2q^3(55m^2+76mn+76n^2),\\
a_3 &= (128m^2+128mn+248n^2)p^4+8q(32m^2+38mn+65n^2)p^3\\
&\quad +3q^2(92m^2+116mn+185n^2)p^2+2q^3(74m^2+119mn+158n^2)p\\
&\quad +11q^4(m+2n)^2,\\
a_4 &= 24(p^2+qp+q^2)[(4m^2+4mn+4n^2+12zm+6zn+10z^2)p^3\\
&\quad +q(6m^2+6mn+6n^2+18zm+8zn+15z^2)p^2+q^2(4m^2+4mn\\
&\quad +2n^2+12zm+5zn+9z^2)p+q^3(m+2z)(m+n+z)],
\end{aligned}$$

$$\begin{aligned}
a_5 &= 12p(p+q)(p^2+qp+q^2)(qm+2pm+pn+qn+qz+2pz)(2pm+qm\\
&\quad +pn+4pz+2qz),\\
Q_1 &= (4m+2n+12z)p^2+(-2wm+4qm-wn-qn+12qz)p\\
&\quad +q(7qm-wm+2qn-2wn+12qz).
\end{aligned}$$

Note that $p\ge 0, q\ge 0, m\ge 0, n\ge 0, z\ge 0$, hence $k_1\ge 0, a_i\ge 0, i=1,2,3,4,5$. Since again $w>0$, thus $Q\ge 0$ follows from the identity (2.14).

Case 2. The non-negative real numbers x,y,z satisfy $x\ge z\ge y$.

We set

$$\begin{cases} z=y+m, & (m\ge 0)\\ x=y+m+n & (n\ge 0).\end{cases} \tag{2.15}$$

By putting (2.12) and (2.15) into the expression of Q, we get

$$k_2Q\equiv b_1w^4+b_2w^3+b_3w^2+b_4w+b_5+w^2Q_2^2, \tag{2.16}$$

where

$$\begin{aligned}
k_2 &= 12(p^2+pq+q^2),\\
b_1 &= (95m^2+98mn+95n^2)p^2+2q(49m^2+49mn+46n^2)p\\
&\quad +q^2(95m^2+92mn+92n^2),\\
b_2 &= (268m^2+328mn+268n^2)p^3+6q(65m^2+82mn+69n^2)p^2\\
&\quad +12q^2(32m^2+39mn+34n^2)p+2q^3(55m^2+76mn+76n^2),\\
b_3 &= (248m^2+368mn+248n^2)p^4+8q(59m^2+92mn+65n^2)p^3\\
&\quad +3q^2(161m^2+254mn+185n^2)p^2+2q^3(113m^2+197mn\\
&\quad +158n^2)p+11q^4(m+2n)^2,\\
b_4 &= 24(p^2+qp+q^2)[(8m^2+10mn+4n^2+14ym+6yn+10y^2)p^3\\
&\quad +q(11m^2+14mn+6n^2+20ym+8yn+15y^2)p^2+q^2(4m^2+5mn\\
&\quad +2n^2+11ym+5yn+9y^2)p+q^3(m+2y)(m+n+y)],\\
b_5 &= 12p(p+q)(p^2+qp+q^2)(pm+qm+pn+qn+yq+2yp)(3pm+qm\\
&\quad +pn+2yq+4yp)\\
Q_2 &= (10m+2n+12y)p^2+(wm+7qm-qn-wn+12yq)p\\
&\quad +q(7qm-wm+2qn-2wn+12yq).
\end{aligned}$$

Since $p\ge 0, q\ge 0, m\ge 0, n\ge 0, y\ge 0$, we have $k_2\ge 0, b_i\ge 0, i=1,2,3,4,5$. Therefore, inequality $Q\ge 0$ follows from (2.16) and $w>0$.

Case 3. The non-negative real numbers x,y,z satisfy $y\ge z\ge x$.

We put

$$\begin{cases} z=x+m, & (m\ge 0)\\ y=x+m+n & (n\ge 0).\end{cases} \tag{2.17}$$

By putting relations (2.12) and (2.17) into the expression of Q, it is easy to check the following identity:

$$k_3Q\equiv c_1w^6+c_2w^5+c_3w^4+c_4w^3+c_5w^2+c_6w+c_7+w^2Q_3^2, \tag{2.18}$$

where

$$\begin{aligned}
k_3 &= 21p^2+(22w+20q)p+(q+2w)(3q+4w),\\
c_1 &= 48m^2,\\
c_2 &= 24m(11pm+6qm+px+qx),\\
c_3 &= (623m^2+214xm+95x^2)p^2+14q(48m^2+19xm+7x^2)p\\
&\quad +q^2(168m^2+144xm+95x^2),\\
c_4 &= (828m^2+160mn+64n^2+688xm+96xn+428x^2)p^3\\
&\quad +2q(657m^2+128mn+48n^2+546xm+80xn+315x^2)p^2\\
&\quad +4q^2(159m^2+28mn+8n^2+189xm+28xn+132x^2)p\\
&\quad +2q^3(36m^2+76xm+8xn+71x^2),\\
c_5 &= (696m^2+472mn+184n^2+1048xm+312xn+752x^2)p^4\\
&\quad +4q(357m^2+240mn+90n^2+524xm+162xn+365x^2)p^3\\
&\quad +q^2(995m^2+652mn+216n^2+1706xm+556xn+1259x^2)p^2\\
&\quad +2q^3(114m^2+70mn+20n^2+291xm+96xn+255x^2)p\\
&\quad +xq^4(40m+20n+51x),\\
c_6 &= (402m^2+508mn+190n^2+808xm+384xn+596x^2)p^5\\
&\quad +2q(503m^2+633mn+233n^2+1001xm+481xn+731x^2)p^4\\
&\quad +2q^2(457m^2+565mn+195n^2+971xm+486xn+709x^2)p^3\\
&\quad +2q^3(169m^2+203mn+63n^2+445xm+232xn+339x^2)p^2\\
&\quad +2q^4(18m^2+21mn+6n^2+89xm+46xn+77x^2)p\\
&\quad +6xq^5(2m+n+2x),\\
c_7 &= p(p+q)(21p^2+20qp+3q^2)(pm+pn+qx+2px)(2qm+3pm\\
&\quad +pn+qn+4px+2qx),\\
Q_3 &= (17m+21n+2x)p^2+(13wm+20qm+20qn+22wn-wx+5qx)p\\
&\quad +6q^2m+4w^2m+wxq+8wqm+5xq^2+10qwn+3q^2n+8w^2n.
\end{aligned}$$

Since $w>0, p\ge 0, q\ge 0, m\ge 0, n\ge 0, x\ge 0$, we have $k_3\ge 0, c_i\ge 0, i=1,2,3,4,5,6,7$. From identity (2.18), we see that inequality $Q\ge 0$ holds.

Case 4. The non-negative real numbers x,y,z satisfy $y\ge x\ge z$.

In this case we put

$$\begin{cases} x=z+m, & (m\ge 0)\\ y=z+m+n & (n\ge 0). \end{cases} \tag{2.19}$$

By putting (2.12) and (2.19) into the expression of Q, we now have the identity:

$$k_4Q\equiv d_1w^6+d_2w^5+d_3w^4+d_4w^3+d_5w^2+d_6w+d_7+w^2Q_4^2, \tag{2.20}$$

where

$$\begin{aligned}
k_4 &= 12p^2+(16w+12q)p+8w^2+8wq+5q^2,\\
d_1 &= 48n^2,
\end{aligned}$$

$$
\begin{aligned}
d_2 &= 24n(10pn+5qn+zq),\\
d_3 &= (504n^2+92z^2)p^2+4q(126n^2+23nz+23z^2)p\\
&\quad +q^2(119n^2+46nz+95z^2),\\
d_4 &= (64m^2+64mn+192mz+568n^2+96nz+368z^2)p^3\\
&\quad +4q(24m^2+24mn+72mz+213n^2+74nz+138z^2)p^2\\
&\quad +4q^2(16m^2+16mn+48mz+102n^2+62nz+117z^2)p\\
&\quad +2q^3(8m^2+8mn+24mz+31n^2+34nz+71z^2),\\
d_5 &= (160m^2+160mn+480mz+352n^2+240nz+512z^2)p^4\\
&\quad +8q(40m^2+40mn+120mz+88n^2+73nz+128z^2)p^3\\
&\quad +4q^2(66m^2+66mn+198mz+125n^2+138nz+221z^2)p^2\\
&\quad +4q^3(26m^2+26mn+78mz+37n^2+59nz+93z^2)p\\
&\quad +q^4(16m^2+16mn+48mz+11n^2+38nz+43z^2),\\
d_6 &= (160m^2+160mn+480mz+112n^2+240nz+368z^2)p^5\\
&\quad +8q(50m^2+50mn+150mz+35n^2+76nz+115z^2)p^4\\
&\quad +4q^2(106m^2+106mn+318mz+66n^2+163nz+241z^2)p^3\\
&\quad +2q^3(118m^2+118mn+354mz+58n^2+178nz+263z^2)p^2\\
&\quad +2q^4(36m^2+36mn+108mz+10n^2+51nz+77z^2)p\\
&\quad +10q^5(m+z)(m+n+2z),\\
d_7 &= (p+q)(12p^2+12pq+5q^2)(qm+2pm+pn+2pz+qz)(qm+2pm\\
&\quad +pn+qn+4pz+2qz)p,\\
Q_4 &= (12m+6n+4z)p^2+(12qm+16wm+8wn+5qn-2wz+4qz)p\\
&\quad +8wqm+8w^2m+5q^2m+3wqn+2q^2n+4w^2n+7q^2z-wqz.
\end{aligned}
$$

Because $p \geq 0, q \geq 0, w > 0, m \geq 0, n \geq 0, z \geq 0$, we have $k_4 \geq 0, d_i \geq 0, i = 1,2,3,4,5,6,7$. Therefore, $Q \geq 0$ holds true.

Case 5. The non-negative real numbers x, y, z satisfy $z \geq x \geq y$.

Put

$$
\begin{cases} x = y + m, & (m \geq 0) \\ z = y + m + n & (n \geq 0). \end{cases} \tag{2.21}
$$

By putting (2.12) and(2.21) into the expression of Q and making the calculations, we obtain

$$
k_5 Q = e_1 w^6 + e_2 w^5 + e_3 w^4 + e_4 w^3 + e_5 w^2 + e_6 w + e_7 + w^2 Q_5^2, \tag{2.22}
$$

where

$$
\begin{aligned}
k_5 &= 29p^2+(24w+22q)p+8w^2+8wq+5q^2,\\
e_1 &= 48n^2,\\
e_2 &= 24(11pn+5qn+yp+yq)n,\\
e_3 &= (95y^2+122yn+623n^2)p^2+14q(7y^2+12yn+41n^2)p\\
&\quad +q^2(95y^2+46yn+119n^2),
\end{aligned}
$$

$$\begin{aligned}
e_4 &= (128m^2+224ym+428y^2+384yn+96mn+732n^2)p^3\\
&\quad +2q(88m^2+160ym+64mn+322yn+315y^2+521n^2)p^2\\
&\quad +4q^2(86yn+111n^2+132y^2+44ym+12mn+16m^2)p\\
&\quad +2q^3(24ym+34yn+31n^2+8m^2+8mn+71y^2),\\
e_5 &= (416n^2+792y^2+408m^2+304mn+752ym+640yn)p^4\\
&\quad +8q(100n^2+185y^2+65mn+169ym+157yn+89m^2)p^3\\
&\quad +q^2(503n^2+408m^2+304mn+922yn+1235y^2+960ym)p^2\\
&\quad +2q^3(60m^2+241y^2+172ym+156yn+52mn+65n^2)p\\
&\quad +q^4(16mn+11n^2+48ym+43y^2+16m^2+38yn),\\
e_6 &= (396mn+536m^2+560yn+116n^2+772y^2+1052ym)p^5\\
&\quad +2q(432mn+591m^2+879y^2+131n^2+1180ym+636yn)p^4\\
&\quad +2q^2(361mn+825y^2+570yn+105n^2+1081ym+486m^2)p^3\\
&\quad +2q^3(395y^2+253yn+204m^2+163mn+37n^2+521ym)p^2\\
&\quad +2q^4(41mn+56yn+5n^2+46m^2+133ym+97y^2)p\\
&\quad +10q^5(y+m)(2y+n+m),\\
e_7 &= p(p+q)(29p^2+22pq+5q^2)(pm+qm+2yp+yq)(3pm+2pn+qm\\
&\quad +qn+4yp+2yq),\\
Q_5 &= (29m+12n+10y)p^2+(wy+11wn+24mw+22qm+8qn+7yq)p\\
&\quad +8mw^2+8mwq+5q^2m-wqy+4w^2n+3wqn+7q^2y+2q^2n).
\end{aligned}$$

Note that $p \geq 0, q \geq 0, w > 0, m \geq 0, n \geq 0, y \geq 0$, and $k_5 > 0, e_i \geq 0, i = 1,2,3,4,5,6,7$. Hence $Q \geq 0$ follows from (2.22).

Case 6. The non-negative real numbers x, y, z satisfy $z \geq y \geq x$.

We put

$$\begin{cases} y = x+m, & (m \geq 0)\\ z = x+m+n & (n \geq 0). \end{cases} \tag{2.23}$$

By putting (2.12) and (2.23) into the expression of Q, we have

$$\begin{aligned}
Q &= 8(m^2+mn+n^2)w^4+(24m^2p+16m^2q+22mnp+16mnq+20n^2p\\
&\quad +10n^2q+2xmp+4xmq+4xnp+2xnq)w^3+(29m^2p^2+36m^2pq\\
&\quad +12m^2q^2+24mnp^2+32mnpq+12mnq^2+16n^2p^2+16n^2pq\\
&\quad +3n^2q^2+20xmp^2+26xmpq+20xmq^2+16xnp^2+16xnpq+10xnq^2\\
&\quad +12x^2p^2+12x^2pq+12x^2q^2)w^2+(16m^2p^3+26m^2p^2q+12m^2pq^2\\
&\quad +12mnp^3+20mnp^2q+10mnpq^2+4n^2p^3+6n^2p^2q+2n^2pq^2\\
&\quad +28xmp^3+44xmp^2q+26xmpq^2+4xmq^3+16xnp^3+24xnp^2q\\
&\quad +12xnpq^2+2xnq^3+20x^2p^3+30x^2p^2q+18x^2pq^2+4x^2q^3)w\\
&\quad +(p+q)(mp+2xp+xq)(3mp+2mq+2np+nq+4xp+2xq)p.
\end{aligned} \tag{2.24}$$

Obviously, inequality $Q \geq 0$ holds in this case, too.

Combining the arguments in the above six cases, we deduce that inequality $Q \geq 0$ holds for non-negative real numbers x, y, z and strictly positive real numbers u, v, w. Therefore, inequalities (2.8) and (1.5) are proved. We find out that equality in (2.8) holds if and only if $x = y = z$ and $u = v = w$. Thus the proof of the theorem is complete. □

3. An application

In this section, we use our Theorem 1.1 to establish a new geometric inequality.

Corollary 3.1. *Let P be an interior point of triangle ABC and let D, E, F be the feet of the perpendiculars from P to the sides BC, CA, AB, respectively. Denote the altitudes of pedal triangle DEF by h_d, h_e, h_f. Then*

$$\frac{r_a}{h_d}+\frac{r_b}{h_e}+\frac{r_c}{h_f} \geq 6, \tag{3.25}$$

where r_a, r_b, r_c are the radii of corresponding three excircles of triangle ABC.

Proof. In what follows, we let $R, \triangle$ denote the circumradius and the area of triangle ABC, respectively; $\triangle_p$ denotes the area of the pedal triangle DEF. We also set $PD = r_1, PE = r_2, PF = r_3, PA = R_1, PB = R_2, PC = R_3$. In the

Theorem 1.1 we take $x=\frac{r_1}{a}, y=\frac{r_2}{b}, z=\frac{r_3}{c}$, then

$$\frac{1}{abc}\left(\frac{cr_2+br_3}{s-a}+\frac{ar_3+cr_1}{s-b}+\frac{br_1+ar_2}{s-c}\right)\geq\frac{12}{ar_1+br_2+cr_3}\left(\frac{r_2r_3}{bc}+\frac{r_3r_1}{ca}+\frac{r_1r_2}{ab}\right).$$

Note that by the well known relation $abc=4R\triangle$ and the following identities:

$$ar_1+br_2+cr_3=2\triangle, \tag{3.26}$$

$$\frac{r_2r_3}{bc}+\frac{r_3r_1}{ca}+\frac{r_1r_2}{ab}=\frac{\triangle_p}{\triangle}, \tag{3.27}$$

we get

$$\frac{cr_2+br_3}{s-a}+\frac{ar_3+cr_1}{s-b}+\frac{br_1+ar_2}{s-c}\geq 24\frac{R\triangle_p}{\triangle},$$

and then by using the fact $(s-a)r_a=(s-b)r_b=(s-c)r_c=\triangle$ and the well-known inequality (see [3]):

$$aR_1\geq cr_2+br_3, \tag{3.28}$$

etc., we have

$$aR_1r_a+bR_2r_b+cR_3r_c\geq 24R\triangle_p. \tag{3.29}$$

Note that again

$$EF=\frac{aR_1}{2R}, FD=\frac{bR_2}{2R}, DE=\frac{cR_3}{2R}, h_d=\frac{\triangle_p}{2EF}, h_e=\frac{\triangle_p}{2FD}, h_f=\frac{\triangle_p}{2DE},$$

and so the claimed inequality follows from (3.29) immediately. □

4. Two conjectures

In the last section, we propose two conjectures which have been checked by the computer.

Conjecture 4.1. *For any triangle ABC and arbitrary positive real numbers x, y, z, the following inequality holds*:

$$\frac{(s-b)(s-c)}{y+z}+\frac{(s-c)(s-a)}{z+x}+\frac{(s-a)(s-b)}{x+y}\leq\frac{3(xa^2+yb^2+zc^2)}{4(yz+zx+xy)}. \tag{4.30}$$

If the above inequality holds, then by applying the same procedure as in the proof of corollary, we can get the geometric inequality:

$$\frac{h_d}{r_b+r_c}+\frac{h_e}{r_c+r_a}+\frac{h_f}{r_a+r_b}\leq\frac{3}{4}. \tag{4.31}$$

But the inequality is still not proved. On the other hand, the author thinks the following stronger inequality holds:

Conjecture 4.2. *Let P be an interior point of triangle ABC, then*

$$\frac{h_d}{w_a}+\frac{h_e}{w_b}+\frac{h_f}{w_c}\leq\frac{3}{2}, \tag{4.32}$$

where w_a, w_b, w_c are the corresponding angle-bisectors of triangle ABC.

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