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A simple proof of Gautschi-Kershaw inequality

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ABSTRACT.

The aim of this paper is to give a simple proof of Gautschi-Kershaw inequality.

1. INTRODUCTION

The first Gautschi-Kershaw inequality states that

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s}\quad(x\geq 1,\ 0<s<1)\,, \tag{1.1}$$

where Γ denotes the Euler gamma function. For more informations on the background of this inequality and its applications, see [2]-[23] and all references therein.

The starting point of the history of this inequality can be considered the work of Wendel [24] who proved

$$\left(\frac{x}{x+a}\right)^{1-a}\leq\frac{\Gamma(x+a)}{x^a\Gamma(x)}\leq 1\quad(0<a<1,\ x>0)\,, \tag{1.2}$$

when he was preoccupied to establish the classical asymptotic relation [1, p. 257, Rel. 6.1.46],

$$\lim_{n\to\infty} n^{s-t}\frac{\Gamma(n+t)}{\Gamma(n+s)}=1. \tag{1.3}$$

The inequality (1.2) was rediscovered by Gautschi [3] in the form

$$x^{1-a}\leq\frac{\Gamma(x+1)}{\Gamma(x+a)}\leq(x+a)^{1-a}\,,$$

then Kershaw [7] among other things, established the double inequality (1.1), now known as the first Gautschi-Kershaw inequality. Different proofs of this inequality employ the convolution theorem of Laplace transforms, asymptotic formulas and integral representations of the gamma, psi and polygamma functions, and other analytic methods.

We give here an elementary proof for the case $x=n$ positive integer, using only the variation of some differentiable functions.

2. THE NEW PROOF

In order to prove (1.1), we define the sequences

$$a_n=\ln\Gamma(n+1)-\ln\Gamma(n+s)-(1-s)\ln\left(n+\frac{s}{2}\right)$$

and

$$b_n=\ln\Gamma(n+1)-\ln\Gamma(n+s)-(1-s)\ln\left(n-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right),$$

then remark that (1.1) is equivalent to $a_n>0$ and $b_n<0$. By (1.3), the sequences a_n and b_n converge to zero. In consequence, it suffices to show that a_n is strictly decreasing and b_n is strictly increasing.

First, we have $a_{n+1}-a_n=f(n)\,,$ where

$$f(x)=\ln(x+1)-\ln(x+s)-(1-s)\ln\left(x+1+\frac{s}{2}\right)+(1-s)\ln\left(x+\frac{s}{2}\right),$$

and we prove that $f<0$. As

$$f'(x)=\frac{s(1-s)(2-s)}{(x+1)(x+s)(2x+s)(2x+s+2)}>0,$$

it results that f is strictly increasing. Moreover, $f(\infty)=0$, so $f<0$.

Now let us denote $b_{n+1}-b_n=g(n)\,,$ where

$$g(x)=\ln(x+1)-\ln\left(x+t^2-\frac{1}{4}\right)$$

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$$+\left(\frac{5}{4}-t^2\right)\ln\left(x+t-\frac{1}{2}\right)-\left(\frac{5}{4}-t^2\right)\ln\left(x+t+\frac{1}{2}\right)$$

and $t=\sqrt{s+\frac{1}{4}}$. As $0<s<1$, it follows $\frac{1}{2}<t<\frac{\sqrt{5}}{2}$. We have

$$g'(x)=-\frac{\left(5-4t^2\right)(2t-1)(3-2t)\,x}{(x+1)\left(4x+4t^2-1\right)(2x+2t-1)(2x+2t+1)}<0,$$

so $g'<0$. Finally, g is strictly decreasing with $g(\infty)=0$, so $g>0$ and the conclusion follows.

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