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Some results on integrals computation and applications to logsine and loggamma integrals

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ABSTRACT.
We describe here some methods for computing of some special integrals. The results are applied to the logsine and the loggamma integrals.

1. INTRODUCTION

In this paper we shall give some methods for computing integrals of functions satisfying certain functional equations. Then we will show how we can we use these abstract results to compute some integrals involving the gamma function.

More precisely, we use our formulas for direct computations of loggamma and logsine integrals, which are of great interest in many branches, such as probabilities, statistical physics, or business and economics. See, e.g., [3]-[6].

2. THE RESULTS

We start with the theorem

Theorem 2.1. *Let $f:[0,a]\to\mathbb{R}$ be continuous and such that*

$$f(2z)=b+f(z)+f(a-z)\ \ ,\ \text{for every } z\in\left[0,\frac{a}{2}\right].$$

Then $\int_0^a f(z)\,\mathrm{d}\,z=-ab.$

Proof. With $z=2t$, we have

$$\int_0^a f(z)\,\mathrm{d}\,z=2\int_0^{a/2} f(2t)\,\mathrm{d}\,t=2\int_0^{a/2}[b+f(t)+f(a-t)]\,\mathrm{d}\,t=$$

$$=ab+2\left[\int_0^{a/2} f(t)\,\mathrm{d}\,t+\int_0^{a/2} f(a-t)\,\mathrm{d}\,t\right].$$

Then by the change of variable $u=a-t$ in the last integral, we obtain

$$\int_0^a f(z)\,\mathrm{d}\,z=ab+2\left[\int_0^{a/2} f(t)\,\mathrm{d}\,t+\int_{a/2}^a f(u)\,\mathrm{d}\,u\right]$$

or

$$\int_0^a f(z)\,\mathrm{d}\,z=ab+2\int_0^a f(t)\,\mathrm{d}\,t,\ \text{thus}\ \int_0^a f(z)\,\mathrm{d}\,z=-ab.$$

□

This result can be obviously applied in case of improper integrals when the respective function is defined on a semiclosed interval. Let us consider now the function

$$f:\left(0,\frac{\pi}{2}\right]\to\mathbb{R}\ \ ,\ \ \ f(z)=\ln\sin z.$$

We have

$$f(2z)=\ln\sin 2z=\ln(2\sin z\cos z)=\ln 2+\ln\sin z+\ln\cos z=$$

$$=\ln 2+\ln\sin z+\ln\sin\left(\frac{\pi}{2}-z\right).$$

Now we can apply Theorem 2.1 with $a=\frac{\pi}{2}$ and $\alpha=\ln 2$ to obtain the known integral

$$\int_0^{\pi/2}\ln\sin z\,\mathrm{d}\,z=-\frac{\pi}{2}\ln 2.$$

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Next we can use the relation

$$\int_0^{\pi/2} f(\sin z)\,\mathrm{d}\,z = \int_0^{\pi/2} f(\cos z)\,\mathrm{d}\,z$$

(which can be obtained by the change of variable $z = \frac{\pi}{2} - t$) to derive

$$\int_0^{\pi/2} \ln\sin z\,\mathrm{d}\,z = \int_0^{\pi/2} \ln\cos z\,\mathrm{d}\,z = -\frac{\pi}{2}\ln 2. \tag{2.1}$$

This result can be obtained in a different way and on this occasion we establish the following abstract result:

Theorem 2.2. *Let $f, g : [-a, a] \to \mathbb{R}$ be continuous and such that*

$$f(x) = f\left(\frac{a+x}{2}\right) + f\left(\frac{a-x}{2}\right) - g(x)\ , \quad \text{for every } x \in [-a, a]\,.$$

Then $\int_0^a f(x)\,\mathrm{d}\,x = \int_0^a g(x)\,\mathrm{d}\,x$.

Proof. By integration from 0 to a we obtain

$$\int_0^a f(x)\,\mathrm{d}\,x = \int_0^a f\left(\frac{a+x}{2}\right)\mathrm{d}\,x + \int_0^a f\left(\frac{a-x}{2}\right)\mathrm{d}\,x - \int_0^a g(x)\,\mathrm{d}\,x.$$

In the right hand we make change of variable $\frac{a+x}{2} = t$, respective $\frac{a-x}{2} = z$, so

$$\int_0^a f(x)\,\mathrm{d}\,x = 2\int_{a/2}^a f(t)\,\mathrm{d}\,t + 2\int_0^{a/2} f(z)\,\mathrm{d}\,z - \int_0^a g(x)\,\mathrm{d}\,x$$

or

$$\int_0^a f(x)\,\mathrm{d}\,x = 2\int_0^a f(t)\,\mathrm{d}\,t - \int_0^a g(x)\,\mathrm{d}\,x \Leftrightarrow \int_0^a f(x)\,\mathrm{d}\,x = \int_0^a g(x)\,\mathrm{d}\,x.$$

□

As a consequence, let us give the following

Corollary 2.1. *Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ be given by $f(x) = -\int_0^x \ln\cos t\,\mathrm{d}\,t$. Then*

$$f(x) = 2f\left(\frac{\pi}{4} + \frac{x}{2}\right) - 2f\left(\frac{\pi}{4} - \frac{x}{2}\right) - x\ln 2\ , \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

and then

$$\int_0^{\pi/2} \ln\sin z\,\mathrm{d}\,z = \int_0^{\pi/2} \ln\cos z\,\mathrm{d}\,z = -\frac{\pi}{2}\ln 2.$$

Proof. Let us start from the elementary identity

$$\frac{1}{2}\cos t = \frac{1}{2}\left(\cos^2\frac{t}{2} - \sin^2\frac{t}{2}\right) =$$

$$= \frac{\sqrt{2}}{2}\left(\cos\frac{t}{2} - \sin\frac{t}{2}\right)\cdot\frac{\sqrt{2}}{2}\left(\cos\frac{t}{2} + \sin\frac{t}{2}\right),$$

so

$$\cos t = 2\cos\left(\frac{\pi}{4} + \frac{t}{2}\right)\cos\left(\frac{\pi}{4} - \frac{t}{2}\right).$$

Hence

$$-\ln\cos t = -\ln\cos\left(\frac{\pi}{4} + \frac{t}{2}\right) - \ln\cos\left(\frac{\pi}{4} - \frac{t}{2}\right) - \ln 2$$

or

$$f'(t) = f'\left(\frac{\pi}{4} + \frac{t}{2}\right) + f'\left(\frac{\pi}{4} + \frac{t}{2}\right) - \ln 2. \tag{2.2}$$

Now let us define $\phi : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ by

$$\phi(x) = f(x) - 2f\left(\frac{\pi}{4} + \frac{x}{2}\right) + 2f\left(\frac{\pi}{4} - \frac{x}{2}\right) + x\ln 2\ , \quad \text{for all } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

With (2.2), we have

$$\phi'(x) = f'(x) - f'\left(\frac{\pi}{4} + \frac{x}{2}\right) - f'\left(\frac{\pi}{4} + \frac{t}{2}\right) + \ln 2 = 0,$$

so ϕ is constant. But $\phi(0)=0$, so $\phi\equiv 0$. Now relation (2.2) for f allows us to use Theorem 2.2, with $a=\frac{\pi}{2}$, $g(t)=\ln 2$. We obtain

$$\int_0^{\pi/2} f'(t)\,\mathrm{d}\,t=\int_0^{\pi/2}\ln 2\,\mathrm{d}\,t \Leftrightarrow f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}\ln 2,$$

which is (2.1). □

Further, the integrals in (2.1) are related with the integral

$$\int_0^1 \ln\Gamma(x)\,\mathrm{d}\,x=\frac{1}{2}\ln 2\pi \tag{2.3}$$

($\Gamma(x)$ is the gamma function) for which we prepare the following abstract result:

Theorem 2.3. *Let $f,g:[0,a]\to\mathbb{R}$ be continuous and such that*

$$f(x)+f(a-x)=g(x)\quad,\quad \text{for all } x\in[0,a]\,.$$

Then $\int_0^a f(x)\,\mathrm{d}\,x=\frac{1}{2}\int_0^a g(x)\,\mathrm{d}\,x.$

Proof. By integration from 0 to a we obtain

$$\int_0^a f(x)\,\mathrm{d}\,x+\int_0^a f(a-x)\,\mathrm{d}\,x=\int_0^a g(x)\,\mathrm{d}\,x.$$

By change of variable $a-x=t$ in the second integral we obtain

$$2\int_0^a f(x)\,\mathrm{d}\,x=\int_0^a g(x)\,\mathrm{d}\,x,$$

which is the conclusion. □

In order to use Theorem 2.3, we also use the following relation

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}\quad,\quad \text{for all } x\in(0,1)\,.$$

Thus

$$\ln\Gamma(x)+\ln\Gamma(1-x)=\ln\frac{\pi}{\sin\pi x},$$

so we can apply Theorem 2.3 with $a=1$, $f(x)=\ln\Gamma(x)$, $g(x)=\frac{\pi}{\sin\pi x}$. Indeed,

$$\int_0^1 \ln\Gamma(x)\,\mathrm{d}\,x=\frac{1}{2}\int_0^1\ln\frac{\pi}{\sin\pi x}\,\mathrm{d}\,x=\frac{1}{2}\ln\pi-\frac{1}{2}\int_0^1\ln\sin\pi x\,\mathrm{d}\,x \tag{2.4}$$

where we are focused to compute the integral

$$J=\int_0^1\ln\sin\pi x\,\mathrm{d}\,x.$$

By change of variable $x=\frac{y}{\pi}$ we obtain

$$\pi J=\int_0^\pi \ln\sin y\,\mathrm{d}\,y=\int_0^{\pi/2}\ln\sin y\,\mathrm{d}\,y+\int_{\pi/2}^{\pi}\ln\sin y\,\mathrm{d}\,y,$$

then by change of variable $y=x+\frac{\pi}{2}$ in the last integral we derive

$$\pi J=\int_0^{\pi/2}\ln\sin y\,\mathrm{d}\,y+\int_0^{\pi/2}\ln\cos y\,\mathrm{d}\,y=-\pi\ln 2$$

(according with (2.1)). We established the relations

$$\int_0^1\ln\sin\pi x\,\mathrm{d}\,x=\int_0^1\ln\cos\pi x\,\mathrm{d}\,x=-\ln 2.$$

Now, by substitute in (2.4) we obtain

$$\int_0^1\ln\Gamma(x)\,\mathrm{d}\,x=\frac{1}{2}\ln\pi-\frac{1}{2}(-\ln 2)=\frac{1}{2}\ln 2\pi,\ \text{as we wanted.}$$

The integral (2.3) can be also calculated in a different way. We give the following result:

Theorem 2.4. *Let $f, g : [0,1] \to \mathbb{R}$ be continuous and such that*

$$f(2x) = f(x) + f\left(x + \frac{1}{2}\right) + g(x) \quad , \text{ for every } x \in \left[0, \frac{1}{2}\right]. \tag{2.5}$$

Then $\int_0^1 f(x)\,\mathrm{d}\,x = -2\int_0^{1/2} g(x)\,\mathrm{d}\,x.$

Proof. By integration from 0 to $1/2$, we obtain

$$\int_0^{1/2} f(2x)\,\mathrm{d}\,x = \int_0^{1/2} f(x)\,\mathrm{d}\,x + \int_0^{1/2} f\left(x + \frac{1}{2}\right)\mathrm{d}\,x + \int_0^{1/2} g(x)\,\mathrm{d}\,x.$$

By evidently change of variables, we get

$$\int_0^{1/2} f(2x)\,\mathrm{d}\,x = \frac{1}{2}\int_0^1 f(x)\,\mathrm{d}\,x \text{ and } \int_0^{1/2} f\left(x + \frac{1}{2}\right)\mathrm{d}\,x = \int_{1/2}^1 f(x)\,\mathrm{d}\,x, \text{ so}$$

$$\frac{1}{2}\int_0^1 f(x)\,\mathrm{d}\,x = \int_0^1 f(x)\,\mathrm{d}\,x + \int_0^{1/2} g(x)\,\mathrm{d}\,x \Leftrightarrow \int_0^1 f(x)\,\mathrm{d}\,x = -2\int_0^{1/2} g(x)\,\mathrm{d}\,x.$$

□

In order to apply this result we use the following multiplication formula for gamma function:

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \cdot \Gamma(x)\Gamma\left(x + \frac{1}{2}\right) \quad , \quad \text{for all } x \in (0, \infty). \tag{2.6}$$

By applying the logarithm, we deduce

$$\ln\Gamma(2x) = \ln\Gamma(x) + \ln\Gamma\left(x + \frac{1}{2}\right) + (2x-1)\ln 2 - \frac{1}{2}\ln\pi,$$

which is relation (2.5) from Theorem 2.4 with

$$f(x) = \ln\Gamma(x) \text{ and } g(x) = (2x-1)\ln 2 - \frac{1}{2}\ln\pi.$$

According to Theorem 2.4, we have

$$\int_0^1 \ln\Gamma(x)\,\mathrm{d}\,x = -2\ln 2\int_0^{1/2}(2x-1)\,\mathrm{d}\,x + \int_0^{1/2}\ln\pi\,\mathrm{d}\,x = \frac{1}{2}\ln 2\pi,$$

so we are done.

A more general form of the multiplication formula (2.6) is

$$\Gamma(x)\Gamma\left(x + \frac{1}{n}\right) \cdot \ldots \cdot \Gamma\left(x + \frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}} \cdot n^{\frac{1}{2}-nx} \cdot \Gamma(nx), \tag{2.7}$$

for every $n \in \mathbb{N}$, $n \geq 2$ and $x \in (0, \infty)$. This formula is also called the Gauss multiplication formula. We will use it to obtain two new results.

Theorem 2.5. *For every $n \in \mathbb{N}$, $n \geq 2$ and $a \in (0, \infty)$, we have*

$$\lim_{n\to\infty} \frac{\sqrt[n]{\Gamma(na)}}{n^a} = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(\int_a^{a+1} \ln\Gamma(x)\,\mathrm{d}\,x\right). \tag{2.8}$$

Proof. By using the characterization of the Riemann integral as a limit, we have

$$\int_a^{a+1} \ln\Gamma(x)\,\mathrm{d}\,x = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln\Gamma\left(a + \frac{k}{n}\right) = \lim_{n\to\infty} \ln\sqrt[n]{\prod_{k=0}^{n-1}\Gamma\left(a + \frac{k}{n}\right)} =$$

$$= \ln\left(\lim_{n\to\infty}\sqrt[n]{(2\pi)^{\frac{n-1}{2}} \cdot n^{\frac{1}{2}-na} \cdot \Gamma(na)}\right) = \ln\left(\sqrt{2\pi} \cdot \lim_{n\to\infty}\left(n^{-a} \cdot \sqrt[n]{\Gamma(na)}\right)\right).$$

By considering the exponential we deduce that

$$\sqrt{2\pi} \cdot \lim_{n\to\infty}\left(n^{-a} \cdot \sqrt[n]{\Gamma(na)}\right) = \exp\left(\int_a^{a+1} \ln\Gamma(x)\,\mathrm{d}\,x\right),$$

then the conclusion follows by dividing both sides by $\sqrt{2\pi}$. □

Finally, we give the following interesting relation:

Theorem 2.6. *For every* $n \in \mathbb{N}$, $n \geq 2$ *and* $a \in (0, \infty)$, *we have*

$$\lim_{a\to 0}\left(\lim_{n\to\infty}\frac{\sqrt[n]{\Gamma(na)}}{n^a}\right) = 1.$$

Proof. By taking the limit as $a \to 0$ in (2.8) and using (2.3), we obtain

$$\lim_{a\to 0}\left(\lim_{n\to\infty}\frac{\sqrt[n]{\Gamma(na)}}{n^a}\right) = \frac{1}{\sqrt{2\pi}}\cdot\exp\left(\frac{1}{2}\ln 2\pi\right) = 1.$$

□

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