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Remark regarding two classes of almost contractions with unique fixed point

MĂDĂLINA PĂCURAR

ABSTRACT.
We discuss the equivalence between two classes of almost contractions with unique fixed point, namely those introduced in [Berinde, V. , *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Analysis Forum, **9** (2004), No. 1, 43-53] and, respectively, [Babu, G. V. R., Sandhya, M. L., Kameswari, M. V. R., *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math., **24** (2008), No. 1, 8-12]. Interesting generalizations are possible in view of this equivalence.

1. INTRODUCTION

Let (X,d) be a metric space and $f : X \to X$ an operator. Regarding this operator we shall consider the following generalized contraction conditions:

- there exist two constants $\delta \in [0,1)$ and $L \geq 0$ such that
$$d(f(x),f(y)) \leq \delta d(x,y) + Ld(y,f(x)), \tag{AC}$$
for any $x,y \in X$;
- there exist two constants $\delta_u \in [0,1)$ and $L_u \geq 0$ such that
$$d(f(x),f(y)) \leq \delta_u d(x,y) + L_u d(x,f(x)), \tag{AC-U}$$
for any $x,y \in X$;
- there exist two constants $\delta_B \in [0,1)$ and $L_B \geq 0$ such that
$$\begin{aligned} d(f(x),f(y)) \leq \delta_B d(x,y) + \\ +L_B \min\{d(x,f(x)), d(y,f(y)), d(x,f(y)), d(y,f(x))\}, \end{aligned} \tag{B}$$
for any $x,y \in X$.

Since the number of recent papers referring to one or many of these interesting conditions is increasing, see for example [4], [5], [6]-[10], the aim of this note is to discuss the relationship between them, as it comes out from the original papers in which they were introduced, namely [2] for conditions (AC) and (AC-U), and [1] for condition (B).

On this occasion we also try to clarify some aspects regarding the proper approach of the non-symmetric condition (AC), which still appears unclear in some papers on weak/almost contractions. Symmetric versions of conditions (AC) and (AC-U) are also given.

2. PRELIMINARIES

In the paper [2] V. Berinde introduced a very interesting class of generalized contractions, initially called *weak contractions*:

Definition 2.1 ([2]). Let (X,d) be a metric space. An operator $f : X \to X$ is called **almost contraction** if it satisfies condition (AC).

The following related result was proved in [2] and reasserted in [3]:

Theorem 2.1 ([2]). *Let (X,d) be a complete metric space and $f : X \to X$ an almost contraction with constants $\delta \in [0,1)$ and $L \geq 0$. Then:*

1) *the Picard iteration $\{x_n\}_{n\geq 0}$ defined by*
$$x_{n+1} = f(x_n),\ n \geq 0, \tag{2.1}$$
converges to $x^*(x_0) \in F_f$, the set of fixed points of f, for any $x_0 \in X$;

2) *for any $x \in X$ we have that $d(x,x^*(x)) \leq \dfrac{1}{1-\delta}\, d(x,f(x))$;*

3) *the following estimates hold:*
$$d(x_n,x^*(x_0)) \leq \frac{\delta^n}{1-\delta}\, d(x_0,x_1), n \geq 1;$$
$$d(x_n,x^*(x_0)) \leq \frac{\delta}{1-\delta}\, d(x_{n-1},x_n), n \geq 1.$$

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Aiming to ensure the uniqueness of the fixed point, V. Berinde [2] added condition (AC-U) and obtained:

Theorem 2.2 ([2]). *Let (X, d) be a complete metric space and $f : X \to X$ an almost contraction with constants $\delta \in [0, 1)$, $L \geq 0$, which in addition satisfies (AC-U) with constants $\delta_u \in [0, 1)$, $L_u \geq 0$, respectively. Then:*

1) *f has a unique fixed point, say x^*, that can be approximated by means of the Picard iteration $\{x_n\}_{n\geq 0}$ defined by (2.1) starting from any $x_0 \in X$;*
2) *for any $x \in X$ we have that $d(x, x^*) \leq \dfrac{1}{1-\delta}\, d(x, f(x))$;*
3) *the following estimates hold:*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta}\, d(x_0, x_1), n \geq 1;$$
$$d(x_n, x^*) \leq \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n), n \geq 1.$$

4) *the rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \delta_u d(x_{n-1}, x^*), n \geq 1.$$

Having in view this result, we introduce:

Definition 2.2. Let (X, d) be a metric space. An operator $f : X \to X$ is called **strict almost contraction** if it satisfies both conditions (AC) and (AC-U), with some real constants $\delta \in [0, 1)$, $L \geq 0$ and $\delta_u \in [0, 1)$, $L_u \geq 0$, respectively.

For a more detailed discussion regarding the relationship between almost contractions, respectively strict almost contractions and other known classes of contractions, including several examples, see [9].

In the same paper [2], V. Berinde raised two open problems, the second of which shall be discussed in this paper:

Problem. *Under the assumptions of Theorem* 2.1, *find a contractive type condition different from (AC-U) that ensures the uniqueness of the fixed point of almost contractions.*

An answer was given by G. V. R. Babu, M. L. Sandhya and M. V. R. Kameswari in the recent paper [1], where the concept of *operator satisfying condition (B)* was introduced. In the following we shall prefer to rename this as *(B)-almost contraction*:

Definition 2.3 ([1]). Let (X, d) be a metric space. An operator $f : X \to X$ is called **(B)-almost contraction** if it satisfies condition (B) with some real constants $\delta_B \in [0, 1)$ and $L_B \geq 0$.

In the same paper [1] the following result was proved as Theorem 2.3, which could be easily enriched with some quantitative information similar to Theorem 2.2 above.

Theorem 2.3. *Let (X, d) be a complete metric space and $f : X \to X$ a (B)-almost contraction with constants $\delta_B \in (0, 1)$ and $L_B \geq 0$. Then f has a unique fixed point.*

3. The main result

Our main aim in this section is to prove that the two classes of almost contractions with unique fixed point mentioned above, namely strict almost contractions and (B)-almost contractions, actually coincide.

In this respect we shall start with some remarks, generated by the easy to notice non-symmetry of both conditions (AC) and (AC-U).

Remark 3.1. If (AC) holds with constants $\delta \in [0, 1)$ and $L \geq 0$ for any $x, y \in X$, then its dual

$$d(f(x), f(y)) \leq \delta d(x, y) + L d(x, f(y)) \tag{dAC}$$

holds for any $x, y \in X$, as well, and vice versa. That is, if they hold for any $x, y \in X$, then (AC) and (dAC) are equivalent.

A practical consequence of this equivalence is the fact that in general only one of them needs to be verified. Still a special attention is to be paid *only* if x and y are *fixed*. In such cases (see for example the proof of Proposition 3 in [2]), one has to check both (AC) and (dAC) in order to clarify whether f fulfills (AC) for the particular x and y, or not.

A way to solve this inconvenient non-symmetry is to consider the following:

Lemma 3.1. *Let (X, d) be a metric space. An operator $f : X \to X$ satisfies condition (AC) with constants $\delta \in [0, 1)$ and $L \geq 0$ if and only if it satisfies the equivalent condition*

$$d(f(x), f(y)) \leq \delta d(x, y) + L \min\{d(x, f(y)), d(y, f(x))\}, \tag{eAC}$$

for any $x, y \in X$.

Proof. In the following we shall assume that $L > 0$, as $L = 0$ would reduce the discussion to the trivial case of Banach contractions.

Let us start by supposing f satisfies (AC). By Remark 3.1 it also satisfies (dAC). But (AC) can be written as

$$\frac{1}{L}\left[d(f(x), f(y)) - \delta d(x, y)\right] \leq d(y, f(x)), \tag{3.2}$$

while (dAC) can similarly be written as:

$$\frac{1}{L}\left[d(f(x), f(y)) - \delta d(x, y)\right] \leq d(x, f(y)). \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\frac{1}{L}\left[d(f(x), f(y)) - \delta d(x, y)\right] \leq \min\{d(x, f(y)), d(y, f(x))\}$$

which can be written back exactly as (eAC):

$$d(f(x), f(y)) \leq \delta d(x, y) + L \min\{d(x, f(y)), d(y, f(x))\},$$

for any $x, y \in X$.

Now supposing f satisfied (eAC), it is immediate that it also fulfills (AC). So the two conditions are equivalent. □

Remark 3.2. Despite theoretical advantages of the symmetric condition (eAC), in most of the situations the short non-symmetric form (AC) is preferred for almost contractions, with no loss of generality if properly used.

A similar approach of the also non-symmetric condition (AC-U) is possible. First let us notice that:

Remark 3.3. If (AC-U) holds with constants $\delta_u \in [0, 1)$ and $L_u \geq 0$ for any $x, y \in X$, then its dual

$$d(f(x), f(y)) \leq \delta_u d(x, y) + L_u d(y, f(y)) \tag{dAC-U}$$

holds for any $x, y \in X$, as well, and vice versa. That is, if they hold for any $x, y \in X$, then (AC-U) and (dAC-U) are equivalent.

Using this remark it is easy to prove:

Lemma 3.2. *Let (X, d) be a metric space. An operator $f : X \to X$ satisfies condition (AC-U) with constants $\delta_u \in [0, 1)$ and $L_u \geq 0$ if and only if it satisfies the equivalent condition*

$$d(f(x), f(y)) \leq \delta_u d(x, y) + L_u \min\{d(x, f(y)), d(y, f(x))\}, \tag{eAC-U}$$

for any $x, y \in X$.

Proof. Assuming again that $L > 0$, condition (AC-U) can be written as

$$\frac{1}{L_u}\left[d(f(x), f(y)) - \delta_u d(x, y)\right] \leq d(x, f(x)), \tag{3.4}$$

while (dAC-U) can be written as:

$$\frac{1}{L_u}\left[d(f(x), f(y)) - \delta_u d(x, y)\right] \leq d(y, f(y)). \tag{3.5}$$

From (3.4) and (3.5) we obtain that

$$\frac{1}{L_u}\left[d(f(x), f(y)) - \delta_u d(x, y)\right] \leq \min\{d(x, f(x)), d(y, f(y))\},$$

which leads exactly to (eAC-U):

$$d(f(x), f(y)) \leq \delta_u d(x, y) + L_u \min\{d(x, f(x)), d(y, f(y))\},$$

for any $x, y \in X$.

Supposing f satisfied (eAC-U), it is immediate that it also satisfies (AC-U). Thus these two conditions are equivalent. □

Now let us prove the main result of the paper:

Theorem 3.4. *Let (X, d) be a metric space. An operator $f : X \to X$ is a strict almost contraction if and only if it is a (B)-almost contraction.*

Proof. For the beginning we shall prove that if f is a (B)-almost contraction, then it is also a strict almost contraction. We assume f satisfies (B) with constants $\delta_B \in [0,1)$ and $L_B \geq 0$. Then for any $x, y \in X$ each one of the following holds:

$$d(f(x), f(y)) \leq \delta_B d(x,y) + L_B d(x, f(x)), \tag{3.6}$$
$$d(f(x), f(y)) \leq \delta_B d(x,y) + L_B d(y, f(y)), \tag{3.7}$$
$$d(f(x), f(y)) \leq \delta_B d(x,y) + L_B d(x, f(y)), \tag{3.8}$$
$$d(f(x), f(y)) \leq \delta_B d(x,y) + L_B d(y, f(x)). \tag{3.9}$$

It is easy to see that (3.6) and (3.7) are actually equivalent, as well as (3.8) and (3.9), respectively.
From (3.6) and (3.9) it follows immediately that f satisfies both (AC) and (AC-U), with constants $\delta = \delta_u = \delta_B$ and $L = L_u = L_B$.

Now let us assume that f is a strict almost contraction, that is, it satisfies (AC) with $\delta \in [0,1)$ and $L \geq 0$ and (AC-U) with $\delta_u \in [0,1)$ and $L_u \geq 0$.
Then, by Lemmas 3.1 and 3.2, f also satisfies (eAC) with δ, L and (eAC-U) with δ_u, L_u. Denoting $\delta_B = \max\{\delta, \delta_u\} \in [0,1)$ and $L_B = \max\{L, L_u\} \geq 0$, conditions (eAC) and (eAC-U) imply:

$$d(f(x), f(y)) \leq \delta_B d(x,y) + L_B \min\{d(x, f(y)), d(y, f(x))\} \tag{3.10}$$

and respectively

$$d(f(x), f(y)) \leq \delta_B d(x,y) + L_B \min\{d(x, f(x)), d(y, f(y))\}, \tag{3.11}$$

for any $x, y \in X$. Rewriting (3.10) and (3.11) under the assumption that $L > 0$, we have that

$$\frac{1}{L_B}\left[d(f(x), f(y)) - \delta_B d(x,y)\right] \leq \min\{d(x, f(y)), d(y, f(x))\} \tag{3.12}$$

and respectively

$$\frac{1}{L_B}\left[d(f(x), f(y)) - \delta_B d(x,y)\right] \leq \min\{d(x, f(x)), d(y, f(y))\}. \tag{3.13}$$

Now from (3.12) and (3.13) it follows that

$$\frac{1}{L_B}\left[d(f(x), f(y)) - \delta_B d(x,y)\right] \leq$$
$$\leq \min\{d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\},$$

which is obviously equivalent to

$$d(f(x), f(y)) \leq \delta_B d(x,y)] +$$
$$+L_B \min\{d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\},$$

for any $x, y \in X$, that is, f satisfies condition (B) with $\delta_B = \max\{\delta, \delta_u\}$ and $L_B = \max\{L, L_u\}$. Now the proof is complete. □

4. Conclusion

The classes of operators satisfying conditions (AC) and (AC-U), respectively, are independent (see also [1]). Their intersection, namely the class of strict almost contractions, coincides with the class of operators satisfying condition (B).

The result proved above as Theorem 3.4 does not express a step backward in the study of almost contractions, but a welcome gain, as it enables the study of the interesting and promising class of strict almost contractions introduced in [2] by means of the very inspired and practical condition (B) proposed in [1].

An example of general result for strict almost contractions, that was possible to attain by using the above mentioned equivalence, can be found in [9]. Other generalizations are also possible.

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"BABES-BOLYAI" UNIVERSITY OF CLUJ-NAPOCA
FACULTY OF ECONOMICS AND BUSINESS ADMINISTRATION
DEPARTMENT OF STATISTICS, FORECAST AND MATHEMATICS
T. MIHALI ST. 58-60, 400591 CLUJ-NAPOCA ROMANIA
E-mail address: madalina.pacurar@econ.ubbcluj.ro