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On generalized algebraic structures

ADINA POP and MARIA S. POP

ABSTRACT.
This paper deals with the generalized algebraic structures with local neutral element. We present as well several examples of generalized monoids.

1. PRELIMINARIES

One of the typical mistakes encountered by the students in the case of composition laws arises from a fundamental wrong understanding of the definition of neutral element.

So, often students say that a composition law " $*$ " defined on a set A has identity (neutral element) if

$$\forall\, x \in A,\ \exists\, e \in A;\quad x * e = e * x = x \tag{1.1}$$

instead of the correct definition:

$$\exists\, e \in A,\ \forall\, x \in A;\ x * e = e * x = x. \tag{1.2}$$

Even though the students are explained that if this element e has a neutral attribute, it has to be in the set A and it has to be the same for every $x \in A$, they still use mostly formula (1.1). We think that more examples where the students have to select a composition law with neutral element, and others with local neutral element, represent the solution to understand better this notion.

However, the study of some algebraic structures where property (1.2) is replaced by (1.1) is interesting, from a scientific point of view.

Recently, Molaei M. R. [5], [6], [7] and Hoseini A. [3] defined some generalized groups and generalized rings by using (1.1).

In this paper we define and give some examples of generalized monoids and we prove some properties of them, following the same approach.

Definition 1.1. A *generalized monoid* is a non-empty set A with an associative binary operation "$\cdot$" called multiplication where, for every $x \in A$, there is a unique $e(x) \in A$ such that $x \cdot e(x) = e(x) \cdot x = x$.

We shall call $e(x)$ a local identity (local neutral element). We note that uniqueness of local identity is very important. Consequently, generalized monoids are universal algebras with two operations: one associative binary operation and another one unary operation which satisfy the local identity law.

Definition 1.2. [5] A generalized monoid $(A, \cdot)$ is called a *generalized group* if for every $x \in A$ there exists an $x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e(x)$.

Definition 1.3. A generalized monoid (group) $(A, \cdot)$ is called *normal* if $e(xy) = e(x)e(y)$, for all $x, y \in A$.

2. PROPERTIES OF GENERALIZED MONOIDS

We start from the following results established in [5].

Proposition 2.1. [5] *Let A be a generalized monoid. If $a \in A$ is an invertible element of A, then a has a unique inverse in A.*

This result motivates the notation a^{-1} for the inverse of a. We denote $U(A) = \{x \in A \mid (\exists)x^{-1} : x^{-1}x = x^{-1}x = e(x)\}$, the set of invertible elements of A.

Proposition 2.2. *If A is a generalized monoid and $a \in A$, then $e(e(a)) = e(a)$ and $e(a)$ is idempotent.*

Proof. Let $a \in A$ be given. Then, there is an unique $e(a) \in A$, such that $a = ae(a) = e(a)a$.

Also, we have $e(e(a))e(a) = e(a)$. Multiplying both sides of the last equality on the right by a we obtain:

$$[e(e(a))e(a)]a = e(a)a.$$

Therefore, by associative property we have $e(e(a))[e(a)a] = a$.

Thus, $e(e(a))a = a$. Analogously, we obtain $ae(e(a)) = a$.

The uniqueness of $e(a)$ shows that $e(e(a)) = e(a)$. On the other hand $[e(a)]^2 = e(a)e(a) = e(a) \cdot e(e(a)) = e(a)$. Hence $e(a)$ is idempotent. □

As in the case of generalized groups [6] and [1], it is easy to prove the following:

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Proposition 2.3. *Let $(A,\cdot)$ be a generalized monoid and let $a \in U(A)$. Then:*
a) $e(a^{-1}) = e(a)$;
b) $(a^{-1})^{-1} = a$.

Proposition 2.4. *Let $(A,\cdot)$ be a normal generalized monoid. If $a, b \in U(A)$ such that $e(a)b^{-1} = b^{-1}e(a)$, then $(ab)^{-1} = b^{-1} \cdot a^{-1}$.*

In [6], Molaei shows that an abelian generalized group is a group. It is easy to see that every group is a Molaei's generalized group.

In [2], Araújo and Konieczny prove that the generalized groups are completely simple semigroups, which represent a well known class of semigroups in semigroup theory.

In the special case of the generalized monoids, the notions of submonoid, and homomorphism of generalized monoids can be extended from the theory of universal algebra as follows.

Definition 2.4. Let $(A,\cdot)$ be a generalized monoid. A non-empty subset $B \subseteq A$ is said to be a *generalized submonoid* of A if for any $x, y \in B$ we have $xy \in B$ and for any $x \in B$ we have $e(x) \in B$.

It is easy to prove the following

Theorem 2.1. *Let $\{B_i; i \in I\}$ be a family of generalized submonoids of the generalized monoid $(A,\cdot)$, and $\bigcap_{i\in I} B_i \neq \emptyset$. Then $\bigcap_{i\in I} B_i$ is a generalized submonoid of A.*

Proposition 2.5. *If $(A,\cdot)$ is a generalized monoid and $a \in A$ is fixed, then the set denoted by $A_a = \{x \in A; e(x) = e(a)\}$ is a generalized submonoid of A. In fact A_a is a monoid.*

Proof. For all $x, y \in A_a$ we have $e(x) = e(y) = e(a)$. Therefore

$$(xy)e(a) = x(ye(a)) = x(ye(y)) = xy.$$

Analogously, we have $e(a)xy = (e(a)x)y = (e(x)x)y = xy$.

Thus $e(xy) = e(a)$, that is, $xy \in A_a$. From Proposition 2.2, if $x \in A_a$, we have $e(e(x)) = e(x) = e(a)$. Hence $e(x) \in A_a$. □

Remark 2.1. In a normal generalized monoid A, the set of local identities $E(A) = \{e(x); x \in A\}$ is a generalized submonoid of the generalized monoid A.

Remark 2.2. Note that not every monoid is a generalized monoid. For instance, the set of residue classes modulo 6, $\mathbb{Z}_6$, with multiplication " $\cdot$ " defined by $\widehat{x} \cdot \widehat{y} = \widehat{x \cdot y}$ is a monoid, but it isn't a generalized monoid because $\widehat{3} \cdot \widehat{1} = \widehat{3} \cdot \widehat{3} = \widehat{3} \cdot \widehat{5} = \widehat{3}$.

If, in addition, the monoid $(M,\cdot,1)$ has the set local identities $E(M) = \{1\}$ then it is a generalized monoid.

Example 2.1. Let the set $\mathbb{N}^3$. We define the binary operation " $*$ " as follows

$$(m,n,p) * (m',n',p') = (m, n+n', p').$$

Then $(\mathbb{N}^3, *)$ is a normal generalized monoid where $e((m,n,p)) = (m,0,p)$. Subsets $\mathbb{N} \times 2\mathbb{N} \times \mathbb{N}$ and $\mathbb{N} \times \{0\} \times \mathbb{N}$ are generalized submonoids of the generalized monoid $(\mathbb{N}^3, *)$.

Example 2.2. The set $\mathbb{N} \times \mathbb{N}^* \times \mathbb{N}$ with the binary operation "$\circ$":

$$(m,n,p) \circ (m',n',p') = (m, nn', p')$$

$(\mathbb{N} \times \mathbb{N}^* \times \mathbb{N}, \circ)$ is a normal generalized monoid and $e((m,n,p)) = (m,1,p)$.

We note that any element of the form $(m,0,p)$ must be excluded (because of them local neutral element isn't unique):

$$(m,0,p) \circ (m,n',p) = (m,0,p), \ \forall n' \in \mathbb{N}.$$

Example 2.3. The pair $(\mathbb{Z}^3, *)$, where "$*$" is a binary operation defined in Example 2.1, is a normal generalized group where $(m,n,p)^{-1} = (m,-n,p)$, and $(\mathbb{N}^3, *)$ from Example 2.1 is a submonoid of $(\mathbb{Z}^3, *)$.

Example 2.4. Let the set of matrices $\mathcal{M}_3(\mathbb{N})$. We define the operation "$*$" as follows

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} * \begin{pmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{pmatrix} = \begin{pmatrix} a & b & c \\ d+d' & e+e' & f+f' \\ g' & h' & i' \end{pmatrix}.$$

Then $(\mathcal{M}_3(\mathbb{N}), *)$ is a normal generalized monoid where

$$e\left(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\right) = \begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ g & h & i \end{pmatrix}.$$

The set $\mathcal{M}_3(\mathbb{Z})$ with the binary operation defined as above is a normal generalized group and $\mathcal{M}_3(\mathbb{N})$ is a normal generalized submonoid.

Example 2.5. The set of residue classes modulo 6, $\mathbb{Z}_6$, with operation " $*$ " defined by $\widehat{x} * \widehat{y} = 3\widehat{x} + 4\widehat{y}$ is a normal generalized group in which every element is idempotent.

Remark 2.3. In fact, a generalized monoid which has all its elements idempotent is a generalized group.

3. Results on homomorphism of generalized monoids

Definition 3.5. Let $(A, *)$, $(B, \circ)$ be two generalized monoids. The map $f : A \to B$ is called a *homomorphism of generalized monoids* if for any $x, y \in A$, we have $f(x * y) = f(x) \circ f(y)$.

If f is an injective homomorphism, then we say that f can be embedded in $(B, \circ)$.

If f is a bijective homomorphism, then f is called *isomorphism*.

Remark 3.4. Let $f : A \to B$ be an homomorphism of generalized monoids, by uniqueness of local identity it follows that for any $x \in A$, $f(e_A(x)) = e_B(f(x))$ is a local identity in the generalized monoid B.

Remark 3.5. Let $(A, *)$ and $(B, \circ)$ be two generalized monoids, $f : A \to B$ a homomorphism of generalized monoids and $a \in A$ an invertible element. Then $f(a)$ is invertible and $[f(a)]^{-1} = f(a^{-1})$.

Definition 3.6. Let $f : A \to B$ be a generalized monoid homomorphism and $a \in A$. Then the set which is denoted by $(Ker\, f)_a = \{x \in A,\ f(x) = e_B(f(a))\}$ is called *the kernel at* a. Obviously, $(Ker\, f)_a = \{x \in A; f(x) = f(e_A(a))\}$.

Remark 3.6. Let $(A, \cdot)$ be a normal generalized monoid. Then the map $e : A \to A$, $x \to e(x)$ is an endomorphism of A and $(Ker\, e)_a = A_a$.

Definition 3.7. Let $f : A \to B$ be a generalized monoids homomorphism. Then the set denoted by $Ker\, f = \bigcup_{a \in A} (Ker\, f)_a$ is called the *kernel of* f.

Note that $Ker\, f = E(A)$.

Example 3.6. The map $f : (\mathbb{N}^3, +) \to (\mathbb{N} \times \mathbb{N}^* \times \mathbb{N}, \circ)$, $f((m, n, p)) = (m, 2^n, p)$ is an injective homomorphism of generalized monoids, where

$$(Ker\, f)_{(m,n,p)} = \{(m, o, p)\} \text{ and } Ker\, f = \mathbb{N} \times \{0\} \times \mathbb{N}.$$

Proposition 3.6. *A cancellative generalized monoid is a monoid.*

Proof. Let $x, y \in A$ and $e(x), e(y)$ be the local identities of them. By associative law, we have $xye(xy) = xy = x(ye(y)) = xye(y)$. Because the cancellative law holds, we obtain $y = ye(xy)$ and $e(xy) = e(y)$.

Analogously, by $e(xy)xy = xy = e(x)xy$ it follows that $e(xy) = e(x)$. Thus, for every $x, y \in A$ we have $e(x) = e(y) = e$. □

The construction of generalized group given by Araújo and Konieczny [2] leads to the next result.

Proposition 3.7. *Any monoid $(M, \cdot, 1)$ with $E(M) = \{1\}$ can be embedded in a normal generalized monoid.*

Proof. Let $(M, \cdot, 1)$ be a monoid, I and J be two non-empty sets. On the set $A = I \times M \times J$ we define the binary operation "$*$" as follows:

$$(i_1, m_1, j_1) * (i_2, m_2, j_2) = (i_1, m_1 m_2, j_2).$$

It is easily verified that the operation "$*$" is associative and for every $(i, m, j) \in A$ there exists the local identity $e((i, m, j)) = (i, 1, j)$. Besides, for every $x = (i_1, m_1, j_1) \in A$, $y = (i_2, m_2, j_2) \in A$ we have

$$e(x * y) = e((i_1, m_1 m_2, j_2)) = (i_1, 1, j_2) = (i_1, 1, j_1) * (i_2, 1, j_2) = e(x) * e(y).$$

Therefore, $(A, *)$ is a normal generalized monoid.

Let $(i, j) \in I \times J$ be an arbitrary fixed pair. The set $S_{ij} = \{i\} \times M \times \{j\}$ is a submonoid of $(A, *)$ isomorphic with $(M, \cdot, 1)$ because the map $f : M \to A$, $f(x) = (i, x, j)$, is an injective homomorphism where $f(M) = S_{ij}$. Moreover, $I \times M \times J = \bigcup_{(i,j) \in I \times J} S_{ij}$. □

Proposition 3.8. *If $(A, \cdot)$ is a generalized monoid and $x \in A$ is idempotent, then $e(x) = x$ and $x \in U(A)$.*

Proof. From $x^2 = x$ it results $xx = x = xe(x)$. From uniqueness of local identity of x we have that $x = e(x)$.

So, $xx = x = e(x)$. From uniqueness of inverse of x it results $x^{-1} = x$ and $x \in U(A)$. □

Corollary 3.1. *Let $(A, \cdot)$ be a normal generalized monoid. The set of idempotents denoted by $I(A)$ is a normal generalized group.*

Proof. Let $x, y \in I(A)$. From Proposition 3.8 we have $x = e(x)$ and $y = e(y)$.
Because A is a normal generalized monoid, from $(xy)e(xy) = xy$ we have $(xy)[e(x)e(y)] = xy$. Thus, $(xy)(xy) = xy$ and so $(xy)^2 = xy$ that is $xy \in I(A)$. For every $x \in I(A)$ there is $e(x) = x$ and $[e(x)]^2 = e(x)$. So, $e(x) \in I(A)$.

From Proposition 3.8 we have $x = x^{-1}$ for every $x \in I(A)$. Hence, $I(A)$ is a normal generalized group. □

Remark 3.7. Note that there are non idempotents in a generalized monoid, which have idempotents local neutral elements.

For instance, in the generalized monoid $(\mathbb{N}^3, *)$ by Example 2.1 the set of idempotents is composed only by local identities.

The map $e : \mathbb{N}^3 \to \mathbb{N}^3$ is a non injective endomorphism. Indeed, if $a = (m, n, p)$, then $a \neq a^2$ for $n \neq 0$ and $e(a^2) = e(a)$. The set $A_a \in A/_{\ker e}$ is the equivalence class of partition $A/_{\ker e}$ and $\ker e$ is an equivalence relation on A.

We have the following correspondence theorem.

Proposition 3.9. *Let $(A, *)$ and $(B, \circ)$ be two generalized monoids and $f : A \to B$ an homomorphism.*

a) If M is a generalized submonoid of A, then $f(M)$ is a generalized submonoid of B;

b) If N is a generalized submonoid of B and $f^{-1}(N) \neq \emptyset$, then $f^{-1}(N)$ is a generalized submonoid of A;

c) If A is a normal generalized monoid, the set $X = \{(e_A(a), f(a)) | a \in A\}$ with binary operation "$*$" defined by $(e_A(a), f(a)) * (e_A(b), f(b)) = (e_A(ab), f(ab))$, is a generalized monoid.

Proof. a) If M is a generalized submonoid of A, then for every $a, b \in M$, we have $ab \in M$ and for any $a \in M$ we have $e(a) \in M$.

For every $x, y \in f(M)$ there exist $a, b \in M$ such that $f(a) = x$, $f(b) = y$. So $xy = f(a) \cdot f(b) = f(ab) \in f(M)$. If $a \in M$, then $e(a) \in M$ and, moreover, $f(a) \in f(M)$, $f(e_A(a)) \in f(M)$, that is $e_B(f(a)) \in f(M)$.

b) We have $f^{-1}(N) = \{a \in A | f(a) \in N\}$. If $a, b \in f^{-1}(N)$ it follows that $f(a), f(b) \in N$. Because N is a generalized submonoid of B, we have $f(a) \cdot f(b) = f(ab) \in N$, and so $ab \in f^{-1}(N)$.

Let $a \in f^{-1}(N)$ namely $f(a) \in N$, be an element. Because N is a generalized submonoid of B, we have $e_B(f(a)) = f(e_A(a)) \in N$. Therefore $e_A(a) \in f^{-1}(N)$ and so $f^{-1}(N)$ is a generalized submonoid of A.

c) Let A be a normal generalized monoid, that is $e_A(ab) = e_A(a) \cdot e_A(b)$, for all $a, b \in A$. We have

$$[(e_A(a), f(a)) * (e_A(b), f(b)] * (e_A(c), f(c)) = (e(ab), f(ab)) * (e(c), f(c))$$
$$= (e_A(abc), f(abc)) = (e_A(a), f(a)) * (e_A(bc), f(bc))$$
$$= (e_A(a), f(a)) * [(e_A(b), f(b)) * (e_A(c), f(c))],$$

thus operation "$*$" is associative.
For every $(e_A(a), f(a)) \in X$ we have:

$$(e_A(a), f(a)) * (e_A(e_A(a)), f(e_A(a))$$
$$= (e_A(a \cdot e_A(a)), f(a \cdot e_A(a))) = (e_A(a), f(a))$$

and

$$(e_A(e_A(a), f(e_A(a))) * (e_A(a), f(a))$$
$$= (e_A(e_A(a)a), f(e_A(a)a)) = (e_A(a), f(a)).$$

From uniqueness of local identities $e_A(a)$ and $e_B(f(a)) = f(e_A(a))$ it follows that

$$e((e_A(a), f(a))) = (e_A(e_A(a)), f(e_A(a))) = (e_A(a), f(e_A(a))).$$

□

Proposition 3.10. *Let $(A, *)$, $(B, \circ)$ be two generalized monoids and $f : A \to B$ a generalized homomorphism.*

a) If $a \in A$, then $(Ker\, f)_a$ is a generalized submonoid of A.

b) If f is an injective homomorphism and $a \in A$ then $(Ker\, f)_a = \{e_A(a)\}$.

Proof. Because $f(e_A(a)) = e_B(f(a))$ we have $e_A(a) \in (Ker f)_a$ and so $(Ker f)_a \neq \emptyset$.
For every $x, y \in (Ker f)_a$ we have

$$f(x * y) = f(x) \circ f(y) = e_B(f(a)) \circ e_B(f(a))$$
$$= f(e_A(a)) \circ f(e_A(a)) = f(e_A(a) * e_A(a)) = f(e_A(a)),$$

From Proposition 2.2 we obtain $f(x * y) = f(e_A(a)) = e_B(f(a))$.
Thus $x * y \in (Ker\, f)_a$. Moreover, if $x \in (Ker\, f)_a$ then $f(e_A(x)) = e_B(f(x)) = e_B(f(e_A(a)) = e_B(f(e_A(a)) = f(e_A(e_A(a))$.
From Proposition 2.2 we obtain $f(e_A(x)) = f(e_A(a))$ and so $f(e_A(x)) = e_B(f(a))$ hence $e_A(x) \in (Ker f)_a$.

3) If f is an injective homomorphism and $x \in (Ker\, f)_a$ then $f(x) = e_B(f(a)) = f(e_A(a))$, hence $x = e_A(a)$. We get $(Ker\, f)_a \subseteq \{e_A(a)\}$. Because $f(e_A(a)) = e_B(f(a))$ we have $\{e_A(a)\} \subseteq (Ker\, f)_a$. Therefore $(Ker\, f)_a = \{e_A(a)\}$. □

In particulary, for generalized group we found theorem 3.5 (ii) in [1], namely:

Corollary 3.2. *Let $a \in G$ and $f : G \to H$ be a generalized group homomorphism. If $Ker f$ at a is denoted by $(Ker f)_a = \{x \in G \,|\, f(x) = f(e(a))\}$, then f is monomorphism if and only if $(Ker f)_a = \{e(a)\}$ for all $a \in G$.*

Corollary 3.3. *If A is a normal generalized monoid, then $(Ker\, e)_a = A_a$ is a submonoid of A.*

Proposition 3.11. *Let $(A, \cdot, 1)$ be a monoid with $E(A) = \{1\}$ and $(B, *)$ be a generalized monoid. If $f : A \to B$ is a generalized monoid homomorphism, then the set $B_{f(1)}$ is a monoid and $f(A)$ is a generalized submonoid of $B_{f(1)}$.*

Proof. $B_{f(1)} = \{b \in B | e_B(b) = e_B(f(1))\} = \{b \in B | e_B(b) = f(e_A(1))\}$.
Because the element 1 is an identity in A, then $e_A(1) = 1$ and so

$$B_{f(1)} = \{b \in B \,|\, e_B(b) = f(1)\}.$$

From Proposition 2.5 it results that the set $B_{f(1)}$ is a monoid.
If $a' \in f(A)$, then there is $a \in A$ such that $f(a) = a'$, so

$$a'f(1) = f(a)f(1) = f(a \cdot 1) = f(a) = a'$$

and

$$f(1)a' = f(1)f(a) = f(1 \cdot a) = f(a) = a'.$$

Hence, $e(a') = f(1)$ and so $a' \in B_{f(1)}$.
In conclusion $f(A) \subset B_{f(1)}$. If $a', b' \in f(A)$ then there exist $a, b \in A$ such that $f(a) = a'$, $f(b) = b'$ and $a \cdot b \in A$. We have $a'b' = f(a) \cdot f(b) = f(ab) \in f(A)$.
If $a' \in f(A)$ then $e_B(a') = e_B(f(a)) = f(e_A(a)) = f(1) \in f(A)$.
So, $f(A)$ is a generalized submonoid of $B_{f(1)}$. □

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North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
E-mail address: adina_p_2006@yahoo.com
E-mail address: maria_sanziana@yahoo.com