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Bézier type curves generated by some class of positive linear operators

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ABSTRACT.

In this paper, we will consider the Bézier type curves in which the fundamental Bernstein basis will be replaced with other function sequences. For all the applications the graphic representation will be done for the same ordinates of the control points. In the first example, we consider Bézier type curves generated by fundamental Bernstein polynomials and "classical" nodes $x_{m,k} = \frac{k}{m}$ and also with "changed " nodes $y_{m,k} = \frac{\sqrt{k(k+1)}}{m}$, where $m = 3$ and $k \in \{0, 1, 2, 3\}$. In the second example, we consider Bézier type curves generated by fundamental Bleimann-Butzer-Hahn polynomials and "classical " nodes $x_{m,k} = \frac{k}{m+1-k}$ and also with "changed " nodes $y_{m,k} = \frac{\sqrt{(m^2-1)k}}{m^2(m+1-k)}$, where $m = 3$ and $k \in \{0, 1, 2, 3\}$. In each example, the "classical " and "changed " points are "close". The obtained curves in each example are plotted in the same figure.

1. INTRODUCTION

In this section, we recall some notions and operators which we will use in the paper.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ be the Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}, \tag{1.2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \ldots, m\}$ (see [1], [4] or [8]).

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [3] the sequence of linear positive operators $(L_m)_{m\geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.3}$$

for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \mid f : [0,\infty) \to \mathbb{R}, f$ is bounded and continuous on $[0,\infty)\}$.

In what follows we consider an interval $I \subset \mathbb{R}$ and we shall use the following sets of functions: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \big\{f \mid f : I \to \mathbb{R}, f$ bounded on $I\big\}$, $C(I) = \big\{f \mid f : I \to \mathbb{R}, f$ continuous on $I\big\}$ and $C_B(I) = B(I) \cap C(I)$.

If $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f; \cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f;\delta) = \sup\left\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\right\}. \tag{1.4}$$

2. PRELIMINARIES

For the following construction and the results as well, see [6]. We consider $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$. Let $I, J \subset [0,\infty)$ be intervals with $I \cap J \neq \emptyset$. For any $m \in \mathbb{N}$ and $k \in \{0, 1, ..., p_m\} \cap \mathbb{N}_0$ consider the nodes $x_{m,k} \in I$ and the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$. Let $E(I)$ and $F(J)$ be subsets of the set of real functions defined on I, respectively J, so that the sum

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$$

exists for any $f \in E(I)$, $\varphi_{m,k} \in F(J)$, $k \in \{0, 1, 2, \ldots, p_m\} \cap \mathbb{N}_0$, $x \in J$ and $m \in \mathbb{N}$. For any $x \in I$ consider the functions $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ and $e_i : I \to \mathbb{R}$, $e_i(t) = t^i$ for any $t \in I$, $i \in \{0, 1, 2\}$. In the following, we suppose that for any $x \in I$ we have $\psi_x \in E(I)$ and $e_i \in E(I)$, $i \in \{0, 1, 2\}$.

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For $m \in \mathbb{N}$ let the given operator $L_m : E(I) \to E(J)$ defined by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k}) \tag{2.1}$$

with the property

$$\lim_{m\to\infty} (L_m f)(x) = f(x), \tag{2.2}$$

for any $x \in J$, uniformly on any compact $K \subset I \cap J$, for any $f \in E(I) \cap C(I)$.

Remark 2.1. From (2.2), for the operators $(L_m)_{m\geq 1}$ we have that

$$\lim_{m\to\infty} (L_m e_i)(x) = e_i(x) \tag{2.3}$$

uniformly on any compact $K \subset I \cap J$, $i \in \{0, 1, 2\}$ and

$$\lim_{m\to\infty} (L_m \psi_x^2)(x) = 0 \tag{2.4}$$

uniformly on any compact $K \subset I \cap J$.

Remark 2.2. From Remark 2.1 follows that for any compact $K \subset I \cap J$ there are the sequences $(u_m(K))_{m\geq 1}$, $(v_m(K))_{m\geq 1}$, $(w_m(K))_{m\geq 1}$ depending on K, so that

$$\lim_{m\to\infty} u_m(K) = \lim_{m\to\infty} v_m(K) = \lim_{m\to\infty} w_m(K) = 0 \tag{2.5}$$

uniformly on K and

$$|(L_m e_0)(x) - 1| \leq u_m(K), \tag{2.6}$$

$$|(L_m e_1)(x) - x| \leq v_m(K), \tag{2.7}$$

$$(L_m \psi_x^2)(x) \leq w_m(K), \tag{2.8}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

In the following, for $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ we consider the nodes $y_{m,k} \in I$ so that

$$\alpha_m = \sup_{k\in\{0,1,\cdots,p_m\}\cap\mathbb{N}_0} |x_{m,k} - y_{m,k}| < \infty \tag{2.9}$$

for any $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \alpha_m = 0. \tag{2.10}$$

For $m \in \mathbb{N}$ and $k \in \{0, 1, \cdots, p_m\} \cap \mathbb{N}_0$ we note $\alpha_{m,k} = x_{m,k} - y_{m,k}$. For $m \in \mathbb{N}$ let us define the operator $K_m : E(I) \to F(J)$ by

$$(K_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(y_{m,k}), \tag{2.11}$$

for any $x \in I$.

Theorem 2.1. [6] *For any $f \in E(I) \cap C(I)$ it follows*

$$\lim_{m\to\infty} (K_m f)(x) = f(x) \tag{2.12}$$

uniformly on any compact $K \subset I \cap J$.

Theorem 2.2. [6] *If $f \in E(I \cap J) \cap C(I \cap J)$ then for any $x \in K = [a, b] \subset I \cap J$ and any $m \in \mathbb{N}$ the following inequality*

$$\begin{aligned} &|(K_m f)(x) - f(x)| \leq |f(x)||(L_m e_0(x)) - 1|+ \\ &+ ((L_m e_0)(x) + 1)\omega(f; \delta_{m,x}) \leq M u_m(K) + (2 + u_m(K))\omega(f; \delta_m), \end{aligned} \tag{2.13}$$

holds, where

$\delta_{m,x} = \sqrt{(L_m e_0)(x)[(L_m \psi_x^2)(x) + 2\alpha_m (L_m e_1)(x) + (\alpha_m^2 + 2x\alpha_m)(L_m e_0)(x)]}$,

$\delta_m = \sqrt{(1 + u_m(K))[w_m(K) + 2\alpha_m(b + v_m(K) + (\alpha_m^2 + 2b\alpha_m)(1 + u_m(K))]}$ *and* $M = \sup\{|f(x)| : x \in K\}$.

Corollary 2.1. [6] *If*

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1 \tag{2.14}$$

for any $x \in J$, then for any $f \in E(I \cap J) \cap C(I \cap J)$, any $x \in K = [a, b] \subset I \cap J$ and any $m \in \mathbb{N}$ it follows

$$|(K_m f)(x) - f(x)| \leq 2\omega(f; \delta_{m,x}) \leq 2\omega(f; \delta'_m), \tag{2.15}$$

where $\delta'_m = \sqrt{w_m(K) + 2\alpha_m v_m(K) + \alpha_m^2 + 4b\alpha_m}$.

3. Bézier type curves

Let $I = [a, b] \cap \mathbb{R}$, $a \in \mathbb{R}$ and $b \in \overline{\mathbb{R}}$ and for all $m \in \mathbb{N}$, $k \in \{0, 1, 2, \ldots, p_m\} \cap \mathbb{N}_0$, the application $\varphi_{m,k} : J \to \mathbb{R}$ with the property $\varphi_{m,k}(t) \geq 0$, for all $t \in J$.

For $m \in \mathbb{N}$, let $a = y_{m,0} < y_{m,1} < y_{m,2} < \ldots$ be the nodes from the set I and the application $\eta_m : J \to I$ defined by

$$\eta_m(t) = \sum_{k=0}^{p_m} \varphi_{m,k}(t) y_{m,k}, \tag{3.1}$$

for all $t \in J$.

Let us consider that for the application η_m we have the property: for any $y_{m,p}$, $p \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$, there exists $t_p \in J$ so that $\eta_m(t_p) = y_{m,p}$ and also the applications $\varphi_{m,k}$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ is continuous on J.
The last condition means that the graph of the application η_m passes through the nodes $y_{m,k}$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$.

In the following lines, we consider as known the values of a continuous application $f : I \to \mathbb{R}$ on the sequence of nodes $\left((y_{m,k})_{k\in\{0,1,\ldots,p_m\}\cap\mathbb{N}_0}\right)_{m\in\mathbb{N}}$, that means $f(y_{m,k}) = z_{m,k}$ for any $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ and for any $m \in \mathbb{N}$.

Definition 3.1. Let $m \in \mathbb{N}$. The points $a_{m,k} = (x_{m,k}; z_{m,k}) \in J \times \mathbb{R}$ and $b_{m,k} = (y_{m,k}; z_{m,k}) \in J \times \mathbb{R}$, where $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ are the classical, respectively changed control points of m order.

For more on Bézier curves topic see [2], [5], [7] and [9].

Definition 3.2. Let $m \in \mathbb{N}$. The classical Bézier curve of m-order, respectively L-Bézier curve of m-order, which correspond to control points $a_{m,k}$,respectively $b_{m,k}$, $k \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$ are:

$$B_m(t) = \sum_{k=0}^{p_m} \varphi_{m,k}(t) a_{m,k} = \left(\sum_{k=0}^{p_m} \varphi_{m,k}(t) x_{m,k}; \sum_{k=0}^{p_m} \varphi_{m,k}(t) z_{m,k} \right), \tag{3.2}$$

$$B_m(t) = \sum_{k=0}^{p_m} \varphi_{m,k}(t) b_{m,k} = \left(\sum_{k=0}^{p_m} \varphi_{m,k}(t) y_{m,k}; \sum_{k=0}^{p_m} \varphi_{m,k}(t) z_{m,k} \right), \tag{3.3}$$

for any $t \in J$.

Remark 3.3. Let $m \in \mathbb{N}$ and $p \in \{0, 1, \ldots, p_m\} \cap \mathbb{N}_0$. Taking into account the above relations, it follows:

$$B_m(t_p) = \left(\sum_{k=0}^{p_m} \varphi_{m,k}(t_p) y_{m,k}; \sum_{k=0}^{p_m} \varphi_{m,k}(t_p) z_{m,k} \right) = (y_{m,p}; z'_{m,p}),$$

where $z'_{m,p} = \sum\limits_{k=0}^{p_m} \varphi_{m,k}(t_p) z_{m,k}$.

In the following it will be of interest to compare the point $z'_{m,p}$ with $f(y_{m,p}) = z_{m,p}$ and on the other hand to compute $z_m(t) = \sum\limits_{k=0}^{p_m} \varphi_{m,k}(t) z_{m,k}$. Finally, we will give graphic representations of these L-Bézier curves. For the Bézier graph, the curves will be blue and for the L-Bézier graph they will be red.

Next, we will consider an operator sequence $(L_m)_{m\geq1}$, which verifies (2.1)–(2.8).

In the applications in the last part of this paper, we will consider the operators from Introduction.

Let us consider that the sequence of nodes $\left((y_{m,k})_{k\in\{0,1,\ldots,p_m\}\cap\mathbb{N}_0}\right)_{n\in\mathbb{N}}$ verifies the conditions (2.9)–(2.10). From (3.2) one obtains that the "ordinate" of $B_m(t)$, according to (2.11) is $(K_m f)(t)$, and then, for the sequence of operators $(K_m)_{m\geq1}$ we can apply Theorems 2.1–2.2 and the Corollary 2.1. Theorem 3.3 and 3.4 are immediate consequences of Theorems 2.1–2.2 and Corollary 2.1, these theorems are theorems of approximations and convergence.

Theorem 3.3. *The following convergences*

$$\lim_{m\to\infty} z_m(t) = f(t) \tag{3.4}$$

and

$$\lim_{m\to\infty} \sum_{k=0}^{p_m} \varphi_{m,k}(t) x_{m,k} = \lim_{m\to\infty} \sum_{k=0}^{p_m} \varphi_{m,k}(t) y_{m,k} = t \tag{3.5}$$

are uniform on any compact $K \subset I \cap J$.

Suppose that

$$\sum_{k=0}^{p_m} \varphi_{m,k}(t) = 1 \tag{3.6}$$

for any $t \in J$.

Theorem 3.4. *For any $t \in K = [a, b] \subset I \cap J$ and any $m \in \mathbb{N}$, the inequalities*

$$|z_m(t) - f(t)| \leq 2\omega(f, \delta'_m) \tag{3.7}$$

and

$$\left|\sum_{k=0}^{p_m} \varphi_{m,k}(t) y_{m,k} - t\right| \leq 2\delta'_m \tag{3.8}$$

holds, where $\delta'_m = \sqrt{w_m(K) + 2\alpha_m v_m(K) + \alpha_m^2 + 4b\alpha_m}$.

In the following lines, we will consider $m = 3$, $z_{3,0} = 1$, $z_{3,1} = 2$, $z_{3,2} = 4$ and $z_{3,3} = 3$.

Application 3.1. If $I = J = [0, 1]$, $E(I) = F(J) = C([0, 1])$, $K = [0, 1]$, $p_m = m$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(t) = p_{m,k}(t)$, $t \in [0, 1]$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$, we obtain the Bernstein operators. We consider the nodes $y_{m,k} = \dfrac{\sqrt{k(k+1)}}{m}$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$. In this case, the operators $(K_m)_{m \geq 1}$ have the form

$$z_m(t) = \sum_{k=0}^{m} p_{m,k}(t) f\left(\frac{\sqrt{k(k+1)}}{m}\right),$$

$f \in C([0, 1])$, $t \in [0, 1]$ and $m \in \mathbb{N}$.

In this application, the control points are $a_0 = (0, 1)$, $a_1 = \left(\frac{1}{3}, 2\right)$, $a_2 = \left(\frac{2}{3}, 4\right)$ and $a_3 = (1, 3)$, respectively $b_0 = (0, 1)$, $b_1 = \left(\frac{\sqrt{2}}{3}, 2\right)$, $b_2 = \left(\frac{\sqrt{6}}{3}, 4\right)$ and $b_3 = \left(\frac{2\sqrt{3}}{3}, 3\right)$. Then

$$\begin{aligned} B_3(t) &= (1-t)^3 a_0 + 3t(1-t)^2 a_1 + 3t^2(1-t) a_2 + t^3 a_3 \\ &= \left(t; -4t^3 + 3t^2 + 3t + 1\right), \end{aligned}$$

$$\begin{aligned} B_3(t) &= (1-t)^3 b_0 + 3t(1-t)^2 b_1 + 3t^2(1-t) b_2 + t^3 b_3 \\ &= \left(\left(\sqrt{2} - \sqrt{6} + \frac{2\sqrt{3}}{3}\right) t^3 + \left(\sqrt{6} - 2\sqrt{2}\right) t^2 + \sqrt{2}t; -4t^3 + 3t^2 + 3t + 1\right), \end{aligned}$$

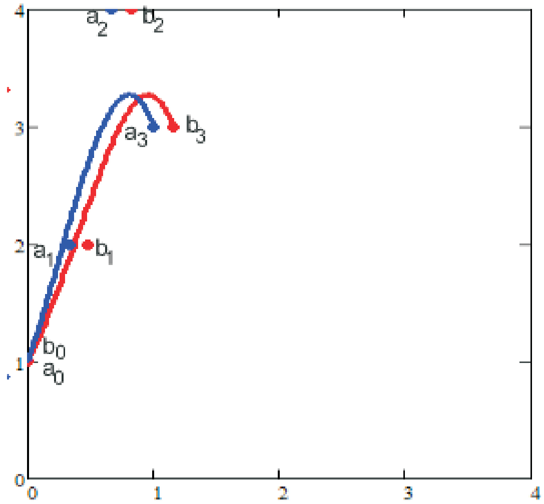
so, the Bézier and the B-Bézier curves in the examples above are given parametrically by

$$\begin{cases} x(t) = t \\ y(t) = -4t^3 + 3t^2 + 3t + 1, \ t \in [0, 1], \end{cases}$$

respectively

$$\begin{cases} x(t) = \left(\sqrt{2} - \sqrt{6} + \dfrac{2\sqrt{3}}{3}\right) t^3 + \left(\sqrt{6} - 2\sqrt{2}\right) t^2 + \sqrt{2}t \\ y(t) = -4t^3 + 3t^2 + 3t + 1, \ t \in [0, 1]. \end{cases}$$

The graphs of these curves are given in the following figure:



Application 3.2. We consider $I = J = [0, \infty)$, $E(I) = F(J) = C_B([0, \infty))$, $K = [0, b]$, $p_m = m$, $x_{m,k} = \dfrac{k}{m+1-k}$, $\varphi_{m,k}(t) = \dfrac{1}{(1+t)^m}\dbinom{m}{k}t^k$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$, $t \in [0, \infty)$. In this case we obtain the Bleimann-Butzer-Hahn operators.

We consider the nodes $y_{m,k} = \dfrac{(m^2-1)k}{m^2(m+1-k)}$, $m \in \mathbb{N}$ and $k \in \{0, 1, \ldots, m\}$. The operators $(K_m)_{m\geq 1}$ have the form

$$z_m(t) = (1+t)^{-m}\sum_{k=0}^{m}\binom{m}{k}t^k f\left(\frac{(m^2-1)k}{m^2(m+1-k)}\right),$$

where $t \in [0, \infty)$, $m \in \mathbb{N}$, $f \in C_B([0, \infty))$.

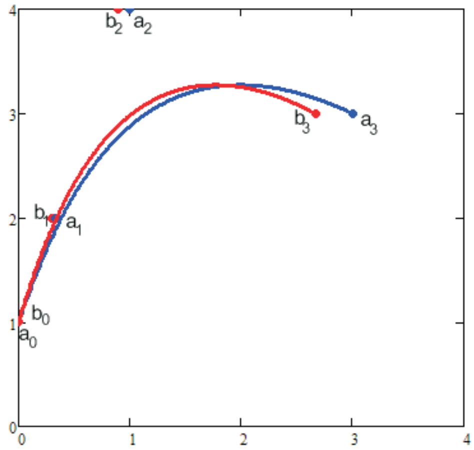
In this case, the control points are: $a_0 = (0, 1)$, $a_1 = \left(\dfrac{1}{3}, 2\right)$, $a_2 = (1, 4)$ and $a_3 = (3, 3)$, respectively $b_0 = (0, 1)$, $b_1 = \left(\dfrac{8}{27}, 2\right)$, $b_2 = \left(\dfrac{8}{9}, 4\right)$ and $b_3 = \left(\dfrac{8}{3}, 3\right)$. Then

$$\begin{cases} x(t) = \dfrac{t + 3t^2 + 3t^3}{(1+t)^3} \\ \\ y(t) = \dfrac{1 + 6t + 12t^2 + 3t^3}{(1+t)^3}, \ t \in [0, \infty), \end{cases}$$

for the Bézier curve and

$$\begin{cases} x(t) = \dfrac{\frac{8}{9}t + \dfrac{8}{3}t^2 + \dfrac{8}{3}t^3}{(1+t)^3} \\ \\ y(t) = \dfrac{1 + 6t + 12t^2 + 3t^3}{(1+t)^3}, \ t \in [0, \infty), \end{cases}$$

for the BBH-Bézier curve. We have the graphical representation of these curves:



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