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Hardy-Hilbert integral inequality with weights

SHANG XIAOZHOU, GAO MINGZHE and MIHÁLY BENCZE

ABSTRACT.

We study a Hardy-Hilbert integral inequality with weights by introducing a weight function of the form $x^{\frac{2}{r}-1}$(with $r > 1$), and prove the constant factor to be the best possible, and give an extension of the Hilbert integral inequality, and then consider some equivalent forms.

1. INTRODUCTION

Let $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. If $f(x) \in L^p(0,+\infty)$ and $g(x) \in L^q(0,+\infty)$, then

$$\int_0^\infty\int_0^\infty \frac{(\ln\frac{x}{y})f(x)g(y)}{x-y}dxdy \le \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.1}$$

and

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \le \frac{\pi}{\sin\frac{\pi}{p}}\left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}. \tag{1.2}$$

where the constant factors $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2$ in (1.1) and $\frac{\pi}{\sin\frac{\pi}{p}}$ in (1.2) are the best possible. And the equalities in (1.1) and (1.2) hold if and only if $f(x)=0$, or $g(x)=0$.They are the famous Hardy-Hilbert integral inequalities, these results can be found in papers [3] and [8]. Owing to the importance of the Hardy-Hilbert inequality in analysis and applications, some mathematicians have been studying them, a great deal of good results are obtained (see [1]-[6], [8]-[10] etc).

It is obvious that the integral kernel function of the left hand side of (1.1) and (1.2) are homogeneous forms of -1-degree. The purpose of the present paper is to study a Hardy-Hilbert integral inequality with a non-homogeneous kernel, and to show that the weight function is $x^{\frac{2}{r}-1}$(with $r>1$), and to prove the constant factor to be the best possible, and to enumerate some important and especial results, and to consider some equivalent forms.

For convenience, we define $\frac{\ln xy}{xy-1}=1$, if $xy=1$.

2. STATEMENT OF MAIN RESULTS

Theorem 2.1. *Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and $f,g\ge 0$. If $\int_0^\infty x^{\frac{2}{q}-1}f^p(x)dx<+\infty$ and $\int_0^\infty x^{\frac{2}{p}-1}g^q(x)dx<+\infty$, then*

$$\int_0^\infty\int_0^\infty \frac{(\ln xy)f(x)g(y)}{xy-1}dxdy \le \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2\left\{\int_0^\infty x^{\frac{2}{q}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{2}{p}-1}g^q(x)dx\right\}^{\frac{1}{q}}, \tag{2.3}$$

where the constant factor $\left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2$ in (2.3) is the best possible. And the equality in (2.3) holds if and only if $f(x)=0$, or $g(x)=0$.

In particular, when $p=2$, we have the following result.

Corollary 2.1. *If $f(x),\ g(x) \in L^2(0,+\infty)$, then*

$$\int_0^\infty\int_0^\infty \frac{(\ln xy)f(x)g(y)}{xy-1}dxdy \le \pi^2\left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}, \tag{2.4}$$

where the constant factor π^2 in (2.4) is the best possible. And the equality in (2.4) holds if and only if $f(x)=0$, or $g(x)=0$.

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Theorem 2.2. *With the assumptions as Theorem 2.1, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{1+xy}dxdy \le \frac{\pi}{\sin(\frac{2}{pq}\pi)}\left\{\int_0^\infty x^{\frac{2}{q}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{\frac{2}{p}-1}g^q(x)dx\right\}^{\frac{1}{q}}, \tag{2.5}$$

where the constant factor $\frac{\pi}{\sin(\frac{2}{pq}\pi)}$ *in* (2.5) *is the best possible. And the equality in* (2.5) *holds if and only iff* $f(x)=0$*, or* $g(x)=0$*.*

Similarly, when $p=2$, we have the following result.

Corollary 2.2. *If* $f(x),\ g(x)\ \in\ L^2(0,\ +\infty)$*, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{1+xy}dxdy \le \pi\left\{\int_0^\infty f^2(x)dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty g^2(x)dx\right\}^{\frac{1}{2}}, \tag{2.6}$$

where the constant factor π *in* (2.6) *is the best possible. And the equality in* (2.6) *holds if and only if* $f(x)=0$*, or* $g(x)=0$*.*

3. Proofs of main results

In order to prove our main results, we need the following lemmas.

Lemma 3.1. *Let* $Rea > Reb > 0$*. Then*

$$\int_{-\infty}^\infty \frac{xe^{bx}}{e^{ax}-1}dx = \left(\frac{\pi}{a\sin\frac{b\pi}{a}}\right)^2. \tag{3.7}$$

This result has been given in the paper [7] (p. 230, formula 1118).

Lemma 3.2. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$*. Then*

$$\int_0^\infty \frac{\ln u}{u-1}\left(\frac{1}{u}\right)^{\frac{2}{pq}}du = \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2. \tag{3.8}$$

Proof. Applying Lemma 1.1 to compute the integral of the left hand side of (3.8) as follows: Substituting e^t for u, it is easy to deduce that

$$\int_0^\infty \frac{\ln u}{u-1}\left(\frac{1}{u}\right)^{\frac{2}{pq}}du = \int_{-\infty}^{+\infty}\frac{te^{(1-\frac{2}{pq})t}}{e^t-1}dt = \left(\frac{\pi}{\sin(1-\frac{2}{pq})\pi}\right)^2 = \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2.$$

□

Lemma 3.3. *Let* $\frac{1}{p}+\frac{1}{q}=1$ *and* $p>1$*. Then*

$$\int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{2}{pq}}du = \frac{\pi}{\sin(\frac{2}{pq}\pi)}. \tag{3.9}$$

Proof. Let $Rem > 0$ and $Ren > 0$. Then the beta function is defined by

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}dt.$$

Substituting $\left(\frac{1}{t}-1\right)$ for u, it is easy to deduce that

$$\int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{2}{pq}}du = \int_0^1 t^{\frac{2}{pq}-1}(1-t)^{-\frac{2}{pq}}dt = B\left(\frac{2}{pq},\ 1-\frac{2}{pq}\right) = \frac{\pi}{\sin(\frac{2}{pq}\pi)}.$$

□

Proof of Theorem 2.1. Let $a = \dfrac{2}{pq}$. Then we may apply the method of the paper [11] and the Hölder inequality to estimate the left-hand side of (2.3) as follows.

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{(\ln xy)f(x)g(y)}{xy-1}dxdy \\
&= \int_0^\infty\int_0^\infty \left(\frac{\ln xy}{xy-1}\right)^{\frac{1}{p}}\left(\frac{x^{a/q}}{y^{a/p}}f(x)\right)\left(\frac{\ln xy}{xy-1}\right)^{\frac{1}{q}}\left(\frac{y^{a/p}}{x^{a/q}}g(y)\right)dxdy \\
&\le \left\{\int_0^\infty\int_0^\infty \frac{\ln xy}{xy-1}\left(\frac{x^{a(p-1)}}{y^a}\right)f^p(x)dxdy\right\}^{\frac{1}{p}}\left\{\int_0^\infty\int_0^\infty \frac{\ln xy}{xy-1}\left(\frac{y^{a(q-1)}}{x^a}\right)g^q(y)dxdy\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\infty \omega(x)x^{ap-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega(y)y^{aq-1}g^q(y)dy\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\infty \omega(x)x^{\frac{2}{q}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega(y)y^{\frac{2}{p}-1}g^q(y)dy\right\}^{\frac{1}{q}},
\end{aligned} \tag{3.10}$$

where $\omega(x) = \int_0^\infty \frac{\ln xy}{xy-1}\left(\frac{x^{1-a}}{y^a}\right)dy$.

Substituting u for xy and then using (3.8), it is easy to deduce that

$$\begin{aligned}
\omega(x) &= \int_0^\infty \frac{\ln xy}{xy-1}\left(\frac{x^{1-a}}{y^a}\right)dy = \int_0^\infty \frac{\ln u}{u-1}\left(\frac{1}{u}\right)^a du \\
&= \left(\frac{\pi}{\sin(a\pi)}\right)^2 = \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2.
\end{aligned} \tag{3.11}$$

It is known from (3.4) and (3.5) that the inequality (2.3) is valid.

If $f(x) = 0$, or $g(x) = 0$, then the equality in (2.3) obviously holds. If $f(x) \neq 0$ and $g(x) \neq 0$, then $0 < \int_0^\infty x^{\frac{2}{q}-1}f^p(x)dx < +\infty$ and $0 < \int_0^\infty x^{\frac{2}{p}-1}g^q(x)dx < +\infty$. If (3.4) takes the form of the equality, then there exists a pair of non-zero constants c_1 and c_2 such that

$$c_1\frac{\ln xy}{xy-1}f^p(x)\left(\frac{x^{a(p-1)}}{y^a}\right) = c_2\frac{\ln xy}{xy-1}g^q(y)\left(\frac{y^{a(q-1)}}{x^a}\right) \quad \text{a.e. on } (0,+\infty)\times(0,+\infty)$$

Then we have

$$c_1x^{ap}\ f^p(x) = c_2\ y^{aq}g^q(y) = C_0 \quad \text{(constant)} \quad \text{a.e. on } (0,+\infty)\times(0,+\infty).$$

Without losing the generality, we suppose that $c_1 \neq 0$, then

$$\int_0^\infty x^{\frac{2}{q}-1}f^p(x)dx = \frac{C_0}{c_1}\int_0^\infty \frac{1}{x}\ dx.$$

This contradicts that $0 < \int_0^\infty x^{\frac{2}{q}-1} f^p(x)dx < +\infty$. Hence it is impossible to take the equality in (3.4). It shows that it is also impossible to take the equality in (2.3).

It remains to need only to show that the constant factor $\left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2$ in (2.3) is the best possible.

Let $a = \dfrac{2}{pq}$. $\forall n \in N$, define two functions by

$$f_n(x) = \begin{cases} x^{-a+\frac{1}{np}}, & x \in (0,\ 1) \\ 0, & x \in [1,\ \infty) \end{cases} \quad \text{and} \quad g_n(y) = \begin{cases} 0, & y \in (0,\ 1) \\ y^{-a-\frac{1}{nq}}. & y \in [1,\ \infty) \end{cases}$$

Then we have

$$\left(\int_0^1 x^{\frac{2}{q}-1}f_n^p(x)dx\right)^{\frac{1}{p}} = n^{\frac{1}{p}}, \quad \left(\int_1^\infty y^{\frac{2}{p}-1}g_n^q(y)dy\right)^{\frac{1}{q}} = n^{\frac{1}{q}}. \tag{3.12}$$

Let $0 < k \leq \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2$ such that the inequality (2.3) is still valid when $\left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2$ is replaced by k. Based on (2.3) and (3.12) we have

$$\frac{1}{n}\int_0^\infty\int_0^\infty \frac{(\ln xy) f_n(x) g_n(y)}{xy\ -1} dxdy \leq \frac{k}{n}\left\{\int_0^\infty x^{\frac{2}{q}-1} f_n^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{\frac{2}{p}-1} g_n^q(y)dy\right\}^{\frac{1}{q}} = k. \tag{3.13}$$

Let $k(1, xy) = \dfrac{\ln xy}{xy-1}$. By Fubini's theorem, it is known from (3.13) that

$$\begin{aligned}
k &\geq \frac{1}{n}\int_0^\infty\int_0^\infty k(1,xy) f_n(x) g_n(y) dxdy = \frac{1}{n}\int_1^\infty y^{-a-\frac{1}{nq}}\left(\int_0^1 k(1,xy)x^{-a+\frac{1}{np}}dx\right)dy \\
&= \frac{1}{n}\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_0^y k(1,u)u^{-a+\frac{1}{np}}du\right)dy \\
&= \frac{1}{n}\left\{\int_1^\infty y^{-1-\frac{1}{n}}\left(\int_0^1 k(1,u)u^{-a+\frac{1}{np}}du\right)dy + \int_1^\infty y^{-1-\frac{1}{n}}\left(\int_1^y k(1,u)u^{-a+\frac{1}{np}}du\right)dy\right\} \\
&= \frac{1}{n}\left\{\int_1^\infty n\left(\int_0^1 k(1,u)u^{-a+\frac{1}{np}}du\right) + \int_1^\infty k(1,u)u^{-a+\frac{1}{np}}\left(\int_u^\infty y^{-1-\frac{1}{n}}dy\right)du\right\} \\
&= \int_0^1 k(1,u)u^{-a+\frac{1}{np}}du + \int_1^\infty k(1,u)u^{-a-\frac{1}{nq}}du.
\end{aligned} \tag{3.14}$$

By Fatou's lemma, we have

$$\begin{aligned}
k &\geq \varliminf_{n\to\infty}\int_0^1 k(1,u)u^{-a+\frac{1}{np}}du + \varliminf_{n\to\infty}\int_1^\infty k(1,u)u^{-a-\frac{1}{nq}}du \\
&\geq \int_0^1 \varliminf_{n\to\infty} k(1,u)u^{-a+\frac{1}{np}}du + \int_1^\infty \varliminf_{n\to\infty} k(1,u)u^{-a-\frac{1}{nq}}du \\
&= \int_0^1 k(1,u)u^{-a}du + \int_1^\infty k(1,u)u^{-a}du \\
&= \int_0^\infty k(1,u)u^{-a}du = \int_0^\infty \frac{\ln u}{u-1}u^{-\frac{2}{pq}}du = \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2.
\end{aligned}$$

The lattermost equality holds based on (3.8).

It follows that $k = \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^2$ in (2.3) is the best possible. Thus the proof of Theorem is completed.

Proof of Theorem 2.2. We assume still that $a = \dfrac{2}{pq}$. Then we may apply the method of the paper [11] and the Hölder inequality to estimate the left-hand side of (2.5) as follows.

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{1+xy}dxdy \\
&= \int_0^\infty\int_0^\infty \left(\frac{1}{1+xy}\right)^{\frac{1}{p}}\left(\frac{x^{a/q}}{y^{a/p}}f(x)\right)\left(\frac{1}{1+xy}\right)^{\frac{1}{q}}\left(\frac{y^{a/p}}{x^{a/q}}g(y)\right)dxdy \\
&\le \left\{\int_0^\infty\int_0^\infty \frac{1}{1+xy}\left(\frac{x^{a(p-1)}}{y^a}\right)f^p(x)dxdy\right\}^{\frac{1}{p}}\left\{\int_0^\infty\int_0^\infty \frac{1}{1+xy}\left(\frac{y^{a(q-1)}}{x^a}\right)g^q(y)dxdy\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\infty \tilde{\omega}(x)x^{ap-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \tilde{\omega}(y)y^{aq-1}g^q(y)dy\right\}^{\frac{1}{q}} \\
&= \left\{\int_0^\infty \tilde{\omega}(x)x^{\frac{2}{q}-1}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \tilde{\omega}(y)y^{\frac{2}{p}-1}g^q(y)dy\right\}^{\frac{1}{q}}, \qquad (3.15)
\end{aligned}$$

where $\tilde{\omega}(x) = \int_0^\infty \frac{1}{1+xy}\left(\frac{x^{1-a}}{y^a}\right)dy$.

Substituting u for xy and then using (3.9), we have

$$\begin{aligned}
\tilde{\omega}(x) &= \int_0^\infty \frac{1}{1+xy}\left(\frac{x^{1-a}}{y^a}\right)dy = \int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^a du \\
&= \int_0^\infty \frac{1}{1+u}\left(\frac{1}{u}\right)^{\frac{2}{pq}} du = \frac{\pi}{\sin(\frac{2}{pq}\pi)}. \qquad (3.16)
\end{aligned}$$

It is known from (3.15) and (3.16) that the inequality (2.5) is valid.

The proof of the rest is similar to one of Theorem 2.1, it is omitted here.

4. Some Equivalent Forms

As applications, we will build some equivalent forms.

Theorem 4.3. *Let* $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ *and* $f \ge 0$. *If* $\int_0^\infty x^{1-\frac{2}{p}}f^p(x)dx < +\infty$, *then*

$$\int_0^\infty y^{(1-\frac{2}{p})(p-1)}\left\{\int_0^\infty \frac{\ln(xy)}{xy-1}f(x)dx\right\}^p dy \le \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^{2p}\int_0^\infty x^{1-\frac{2}{p}}f^p(x)dx, \qquad (4.17)$$

where the constant factor $\left(\frac{\pi}{\sin(\frac{2}{pq}\pi)}\right)^{2p}$ *is the best possible, and the equality in* (4.17) *holds if and only if* $f(x) = 0$, *and inequality* (4.17) *is equivalent to* (2.3).

Proof. Let $g(y) = (y^{(1-\frac{2}{p})} \int_0^\infty \frac{\ln(xy)}{xy-1} f(x)dx)^{p-1}$. Then by (2.3), we have

$$\int_0^\infty y^{(1-\frac{2}{p})(p-1)} \left\{ \int_0^\infty \frac{\ln(xy)}{xy-1} f(x)dx \right\}^p dy = \int_0^\infty \int_0^\infty \frac{\ln(xy)}{xy-1} f(x)g(y)dxdy$$

$$\leq \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^2 \left\{ \int_0^\infty x^{\frac{2}{q}-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{\frac{2}{p}-1} g^q(y)dy \right\}^{\frac{1}{q}}$$

$$= \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^2 \left\{ \int_0^\infty x^{1-\frac{2}{p}} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{\frac{2}{p}-1} g^q(y)dy \right\}^{\frac{1}{q}}$$

$$= \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^2 \left\{ \int_0^\infty x^{1-\frac{2}{p}} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{(1-\frac{2}{p})(p-1)} \left(\int_0^\infty \frac{\ln(xy)}{xy-1} f(x)dx \right)^p dy \right\}^{\frac{1}{q}}. \tag{4.18}$$

The inequality (4.17) follows from (4.18) after some simplifications.

On the other hand, assume that the inequality (4.17) is valid. Apply in turn the Hölder inequality and (4.17), we have

$$\int_0^\infty \int_0^\infty \frac{\ln(xy)}{xy-1} f(x)g(y)dxdy$$

$$= \int_0^\infty \left\{ y^{\frac{1}{p}(1-\frac{2}{p})(p-1)} \int_0^\infty \frac{\ln(xy)}{xy-1} f(x)dx \right\} \left\{ y^{-\frac{1}{p}(1-\frac{2}{p})(p-1)} g(y)dy \right\}$$

$$\leq \left\{ \int_0^\infty y^{(1-\frac{2}{p})(p-1)} \left(\int_0^\infty \frac{\ln(xy)}{xy-1} f(x)dx \right)^p dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{\frac{2}{p}-1} g^q(y)dy \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^{2p} \int_0^\infty x^{1-\frac{2}{p}} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{\frac{2}{p}-1} g^q(y)dy \right\}^{\frac{1}{q}}$$

$$= \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^2 \left\{ \int_0^\infty x^{\frac{2}{q}-1} f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{\frac{2}{p}-1} g^q(y)dy \right\}^{\frac{1}{q}}. \tag{4.19}$$

If the constant factor $\left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^{2p}$ in (4.17) is not the best possible, then it is known from (4.19) that the constant factor $\left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^2$ in (2.3) is also not the best possible, this is in contradiction. Evidently, the equality in (4.17) holds if and only if $f(x) = 0$. Consequently, the inequality (4.17) is equivalent to (2.3). The proof of Theorem is completed. □

Theorem 4.4. *With the assumptions as Theorem 2.1, then*

$$\int_0^\infty y^{(1-\frac{2}{p})(p-1)} \left\{ \int_0^\infty \frac{1}{1+xy} f(x)dx \right\}^p dy \leq \left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^p \int_0^\infty x^{1-\frac{2}{p}} f^p(x)dx, \tag{4.20}$$

where the constant factor $\left(\frac{\pi}{\sin(\frac{2}{pq}\pi)} \right)^p$ *is the best possible, and the equality in* (4.20) *holds if and only if* $f(x) = 0$. *Inequality* (4.20) *is equivalent to* (2.5).

The proof of Theorem 4.4 is similar to one of Theorem 4.3, it is omitted here.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
NORMAL COLLEGE OF JISHOU UNIVERSITY
JISHOU, HUNAN, 416000, P. R. CHINA
E-mail address: mingzhegao@163.com
E-mail address: jdsysxz003@163.com

HĂRMANULUI 6, 505600 SĂCELE, ROMANIA
E-mail address: benczemihaly@gmail.com