CREATIVE MATH. & INF.
19 (2010), No. 2, 227 - 234

Online version available at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Some new extensions on Hilbert's integral inequality and its applications

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ABSTRACT.

We give some new extensions and refinements of Hilbert's integral inequality with parameters λ and $\mu\Big(0<\mu\le 2, 0<\frac{\mu}{\lambda}<2\Big)$ by introducing a proper weight function. As applications, some extensions and strengthened results of Widder's theorem and Hardy- Littlewood's theorem are also given.

1. INTRODUCTION AND LEMMAS

Let $f(x),\ g(x)\ \in\ L^2(0,\ +\infty)$. It is all known that the inequality of the form

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)g(y)}{x+y-2\alpha}\mathrm{d}x\mathrm{d}y\ \le\pi\left\{\int_{\alpha}^{\infty}f^2(x)\mathrm{d}x\right\}^{\frac12}\left\{\int_{\alpha}^{\infty}g^2(x)\mathrm{d}x\right\}^{\frac12}\tag{1.1}$$

is called Hilbert's integral inequality, where the constant factor π is the best possible.

In the paper [6], by introducing a parameter $\lambda\ \Big(\lambda>\frac12\Big)$, the extension of the form

$$\int_{0}^{\infty}\int_{0}^{\infty}\frac{f(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y\le\Big(\frac{\pi}{\lambda\sin\frac{\pi}{2\lambda}}\Big)\left\{\int_{0}^{\infty}x^{1-\lambda}f^2(x)\mathrm{d}x\right\}^{\frac12}\left\{\int_{0}^{\infty}x^{1-\lambda}g^2(x)\mathrm{d}x\right\}^{\frac12}\tag{1.2}$$

was established.

Recently, various improvements and extensions of (1.1) appear in a great deal of papers (see [2]). The aim of this paper is to give some new extensions and improvements of (1.1) and (1.2) and then to give its some applications, and the method adopted by us has trait itself, it is different from those listed in the paper [2]. Explicitly, the idea and the results obtained possess new meanings.

In the sake of convenience, we define a function and introduce some notations.

Let $0<\mu\le 2, 0<\frac{\mu}{\lambda}<2, x-\alpha\ge 0, y-\alpha\ge 0$, $c(x)$ be an integrable function in $[0,+\infty)$.

Define a binary function by $E(x,y)=1-c(x-\alpha)+c(y-\alpha)$, and we stipulate that $E(x,y)\ge 0$ for $(x,y)\in(0,+\infty)\times(0,+\infty)$. And we introduce also the following notations:

$$A_1=\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f^2(x)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\Big(\frac{x-\alpha}{y-\alpha}\Big)^{1-\frac{\mu}{2}}E(x,y)\mathrm{d}x\mathrm{d}y$$

$$A_2=\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f^2(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\Big(\frac{y-\alpha}{x-\alpha}\Big)^{1-\frac{\mu}{2}}E(x,y)\mathrm{d}x\mathrm{d}y.$$

Throughout the paper we use frequently these notations.

In order to prove our assertion, we need the following lemmas.

Lemma 1.1. *With the assumptions as the above-mentioned, then*

$$A_1A_2=\Big(\frac{\pi}{\lambda\sin(\frac{\mu\pi}{2\lambda})}\Big)^2\left\{\Big(\int_{\alpha}^{\infty}(x-\alpha)^{1-\lambda}f^2(x)\mathrm{d}x\Big)^2-\Big(\int_{\alpha}^{\infty}k(x)f^2(x)\mathrm{d}x\Big)^2\right\}\tag{1.3}$$

where the weight function $k(x)$ is defined by

$$k(x)=(x-\alpha)^{1-\lambda}\left\{\Big(\frac{\lambda\sin\frac{\mu\pi}{2\lambda}}{\pi}\Big)\int_{0}^{\infty}\frac{c\,((x-\alpha)t)t^{\frac{\mu}{2}-1}}{1+t^\lambda}\mathrm{d}t-c(x-\alpha)\right\}.\tag{1.4}$$

Received: 21.01.2010; In revised form: 24.07.2010; Accepted: 15.08.2010.
2000 *Mathematics Subject Classification*. 26D15.
Key words and phrases. Weight function, Hilbert's integral inequality, Schwarz's inequality, Widder's theorem, Hardy-Littlewood's theorem.

Proof. It is easy to deduce that

$$A_1 = \int_\alpha^\infty \left\{ \int_\alpha^\infty \frac{1}{(x-\alpha)^\lambda (1+(\frac{y-\alpha}{x-\alpha})^\lambda)} \left(\frac{x-\alpha}{y-\alpha}\right)^{1-\frac{\mu}{2}} E(x,y)\mathrm{d}y \right\} f^2(x)\mathrm{d}x$$
$$= \int_\alpha^\infty \left\{ \int_0^\infty \frac{1}{1\, +\, t^\lambda} \left(\frac{1}{t}\right)^{1-\frac{\mu}{2}} (1 - c(x-\alpha) + c((x-\alpha)t))\mathrm{d}t \right\} (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x$$
$$= \int_\alpha^\infty \left\{ \frac{\pi}{\lambda \sin \frac{\mu\pi}{2\lambda}} + \int_0^\infty \frac{c((x-\alpha)t)}{1+t^\lambda} \left(\frac{1}{t}\right)^{1-\frac{\mu}{2}} \mathrm{d}t - (\frac{\pi}{\lambda \sin \frac{\mu\pi}{2\lambda}})\, c(x-\alpha) \right\} (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x$$
$$= \left(\frac{\pi}{\lambda \sin \frac{\mu\pi}{2\lambda}}\right) \left\{ \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x + \int_\alpha^\infty k(x) f^2(x)\mathrm{d}x \right\},$$

where $k(x)$ is a function defined by (1.4).

Similarly, we have $A_2 = \left(\frac{\pi}{\lambda \sin \frac{\mu\pi}{2\lambda}}\right) \left\{ \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x - \int_\alpha^\infty k(x) f^2(x)\mathrm{d}x \right\}$.

It follows that the relation (1.3) holds. □

Lemma 1.2. *Let* $0 < \mu \le 2, 0 < \frac{\mu}{\lambda} < 2$. *Then*

$$\int_0^\infty \frac{t^{\frac{\mu}{2}-1}}{(1+t^\lambda)(1+(x-\alpha)^\lambda\, t^\lambda)} \mathrm{d}t = \begin{cases} \left(\frac{\pi}{\lambda \sin(\frac{\mu\pi}{2\lambda})}\right) \frac{(x-\alpha)^{\lambda-\frac{\mu}{2}} - 1}{(x-\alpha)^\lambda - 1}, & x-\alpha \ne 1, \\ \left(\frac{\pi}{\lambda \sin(\frac{\mu\pi}{2\lambda})}\right) \left(1 - \frac{\mu}{2\lambda}\right). & x - \alpha = 1. \end{cases}$$

Proof. When $x-\alpha \ne 1$, it has been given in the papers [5] and [8]. Let's consider the case $x-\alpha = 1$. By the definition and properties of beta function, it is easy to deduce that

$$\int_0^\infty \frac{t^{\frac{\mu}{2}-1}}{(1+t^\lambda)^2} \mathrm{d}t = \frac{1}{\lambda} B\left(\frac{\mu}{2\lambda}, 2 - \frac{\mu}{2\lambda}\right) = \frac{1}{\lambda}\left(1 - \frac{\mu}{2\lambda}\right) B\left(\frac{\mu}{2\lambda},\, 1 - \frac{\mu}{2\lambda}\right)$$
$$= \left(1 - \frac{\mu}{2\lambda}\right) \frac{\pi}{\lambda \sin \frac{\mu\pi}{2\lambda}}\,.$$

□

2. Main Results

In this section we shall prove our assertions with the help of the above lemmas.

Theorem 2.1. *Let* $f(x)$ *and* $g(x)$ *be two real functions,* $0 < \mu \le 2$ *and* $0 < \frac{\mu}{\lambda} < 2$. *If* $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x < +\infty$ *and* $0 < \int_\alpha^\infty (x-\alpha)^{1-\lambda} g^2(x)\mathrm{d}x < +\infty$ *then*

$$\left(\int_\alpha^\infty \int_\alpha^\infty \frac{f(x)g(y)}{(x-\alpha)^\lambda + (y-\alpha)^\lambda} \mathrm{d}x\mathrm{d}y\right)^4$$
$$\le \left(\frac{\pi}{\lambda \sin \frac{\mu\pi}{2\lambda}}\right)^4 \left\{ \left(\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x\right)^2 - \left(\int_\alpha^\infty \tilde{\omega}(x) f^2(x)\mathrm{d}x\right)^2 \right\}$$
$$\times \left\{ \left(\int_\alpha^\infty (x-\alpha)^{1-\lambda} g^2(x)\mathrm{d}x\right)^2 - \left(\int_\alpha^\infty \tilde{\omega}(x) g^2(x)\mathrm{d}x\right)^2 \right\} \tag{2.5}$$

where the weight function $\tilde{\omega}(x)$ *is defined by*

$$\tilde{\omega}(x) = \begin{cases} (x-\alpha)^{1-\lambda} \left\{ \frac{(x-\alpha)^{\lambda-\frac{\mu}{2}} - 1}{(x-\alpha)^\lambda - 1} - \frac{1}{1+(x-\alpha)^\lambda} \right\}, & x-\alpha \ne 1, \\ \frac{1}{2} - \frac{\mu}{2\lambda}. & x-\alpha = 1. \end{cases} \tag{2.6}$$

Proof. At first, suppose that $f = g$. Applying Schwarz's inequality we have

$$\Big(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)f(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\mathrm{d}x\mathrm{d}y\Big)^2 = \Big(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)\ f(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}E(x,y)\mathrm{d}x\mathrm{d}y\Big)^2$$

$$= \Big(\int_\alpha^\infty\int_\alpha^\infty\Big\{\frac{f(x)}{((x-\alpha)^\lambda+(y-\alpha)^\lambda)^{1/2}}\Big(\frac{x-\alpha}{y-\alpha}\Big)^{\frac{1}{2}(1-\frac{\mu}{2})}(E(x,y))^{\frac{1}{2}}\Big\}$$

$$\times\Big\{\frac{f(y)}{((x-\alpha)^\lambda+(y-\alpha)^\lambda)^{1/2}}\Big(\frac{y-\alpha}{x-\alpha}\Big)^{\frac{1}{2}(1-\frac{\mu}{2})}(E(x,y))^{\frac{1}{2}}\Big\}\mathrm{d}x\mathrm{d}y\Big)^2$$

$$\le \int_\alpha^\infty\int_\alpha^\infty\Big\{\frac{f^2(x)}{(x-\alpha)^\lambda\ +\ (y-\alpha)^\lambda}(\frac{x-\alpha}{y-\alpha})^{(1-\frac{\mu}{2})}E(x,y)\Big\}\mathrm{d}x\mathrm{d}y$$

$$\times\int_\alpha^\infty\int_\alpha^\infty\Big\{\frac{f^2(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\Big(\frac{y-\alpha}{x-\alpha}\Big)^{(1-\frac{\mu}{2})}E(x,y)\Big\}\mathrm{d}x\mathrm{d}y = A_1A_2.$$

It follows from (1.3) that

$$\Big(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)f(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\mathrm{d}x\mathrm{d}y\Big)^2$$

$$\le\Big(\frac{\pi}{\lambda\sin(\frac{\mu\pi}{2\lambda})}\Big)^2\Big\{\Big(\int_\alpha^\infty(x-\alpha)^{1-\lambda}f^2(x)\mathrm{d}x\Big)^2-\Big(\int_\alpha^\infty k(x)f^2(x)\mathrm{d}x\Big)^2\Big\},\tag{2.7}$$

where the weight function $k(x)$ is defined by (1.4).

Let $c(x) = \dfrac{1}{1+x^\lambda}$. It is obvious that $E(x,y)\ge 0$. It is easy to deduce that

$$k(x) = (x-\alpha)^{1-\lambda}\Big\{\Big(\frac{\lambda\sin\frac{\mu\pi}{2\lambda}}{\pi}\Big)\int_0^\infty\frac{c\left((x-\alpha)t\right)t^{\frac{\mu}{2}-1}}{1+t^\lambda}\mathrm{dt}-\ c(x-\alpha)\Big\}$$

$$= (x-\alpha)^{1-\lambda}\Big\{\Big(\frac{\lambda\sin\frac{\mu\pi}{2\lambda}}{\pi}\Big)\int_0^\infty\frac{t^{\frac{\mu}{2}-1}}{(1+t^\lambda)\ (1+(x-\alpha)^\lambda t^\lambda)}\mathrm{dt}-c(x-\alpha)\Big\}.$$

It follows from Lemma 2.2 that $k(x)=\tilde\omega(x)$. So we have

$$\Big(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)f(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\mathrm{d}x\mathrm{d}y\ \Big)^2$$

$$\le\Big(\frac{\pi}{\lambda\sin\left(\frac{\mu\pi}{2\lambda}\right)}\Big)^2\Big\{\Big(\int_\alpha^\infty(x-\alpha)^{1-\lambda}f^2(x)\mathrm{d}x\Big)^2-\Big(\int_\alpha^\infty\tilde\omega(x)f^2(x)\mathrm{d}x\Big)^2\Big\},\tag{2.8}$$

where $\tilde\omega(x)$is defined by (2.6). Hence the inequality (2.5) is valid.

Let's consider the case $f\ne g$. By Schwarz's inequality we have

$$\Big(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)\ g(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\mathrm{d}x\mathrm{d}y\Big)^4$$

$$=\Big\{\Big(\int_0^1\Big(\int_\alpha^\infty t^{(x-\alpha)^\lambda-\frac{1}{2}}f(x)\mathrm{d}x\int_\alpha^\infty t^{(y-\alpha)^\lambda-\frac{1}{2}}g(y)\mathrm{d}y\Big)\mathrm{dt}\Big)^2\Big\}^2$$

$$\le\Big\{\int_0^1\Big(\int_\alpha^\infty t^{(x-\alpha)^\lambda-\frac{1}{2}}f(x)\mathrm{d}x\Big)^2\mathrm{dt}\Big\}^2\Big\{\int_0^1\Big(\int_\alpha^\infty t^{(y-\alpha)^\lambda-\frac{1}{2}}g(y)\mathrm{d}y\Big)^2\mathrm{dt}\Big\}^2$$

$$=\Big\{\int_\alpha^\infty\int_\alpha^\infty\frac{f(x)\ f(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\mathrm{d}x\mathrm{d}y\Big\}^2\Big\{\int_\alpha^\infty\int_\alpha^\infty\frac{g(x)\ g(y)}{(x-\alpha)^\lambda+(y-\alpha)^\lambda}\mathrm{d}x\mathrm{d}y\Big\}^2.\tag{2.9}$$

Based on (2.8), it follows from (2.9) that the inequality (2.5) is valid at once. Theorem is proved. □

When $\lambda=1$ and $\mu=1$, based on (2.5), the following result is obtained.

Theorem 2.2. *If* $0<\int_\alpha^\infty f^2(x)\mathrm{d}x<+\infty$ *and* $0<\int_\alpha^\infty g^2(x)\mathrm{d}x<+\infty$ *then*

$$\Big(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)g(y)}{x+y-2\alpha}\mathrm{d}x\mathrm{d}y\Big)^4 \le \pi^4\left\{\Big(\int_\alpha^\infty f^2(x)\mathrm{d}x\Big)^2-\Big(\int_\alpha^\infty \omega(x)f^2(x)\mathrm{d}x\Big)^2\right\} \times\left\{\Big(\int_\alpha^\infty g^2(x)\mathrm{d}x\Big)^2-\Big(\int_\alpha^\infty \omega(x)g^2(x)\mathrm{d}x\Big)^2\right\}, \tag{2.10}$$

where the weight function $\omega(x)$ *is defined by*

$$\omega(x)=\frac{1}{\sqrt{x-\alpha}+1}-\frac{1}{x-\alpha+1}. \tag{2.11}$$

Proof. It is known from (2.6) that $\tilde{\omega}(x)$ can be reduced to $\omega(x)$, when $\lambda=\mu=1$. Hence the inequality (2.10) is valid. □

Corollary 2.1. *With the assumption as Theorem 2.2 and* $f=g$ *then*

$$\Big(\int_\alpha^\infty\int_\alpha^\infty \frac{f(x)f(y)}{x+y-2\alpha}\mathrm{d}x\mathrm{d}y\Big)^2 \le \pi^2\left\{\Big(\int_\alpha^\infty f^2(x)\mathrm{d}x\Big)^2-\Big(\int_\alpha^\infty \omega(x)f^2(x)\mathrm{d}x\Big)^2\right\}, \tag{2.12}$$

where the weight function $\omega(x)$ *is defined by* (2.11).

When $c(x)=$ constant, we have $k(x)=\tilde{\omega}(x)=0$. If $\alpha=0$ and $\mu=1$, then the inequality (2.5) can be reduced to the inequality (1.2). Similarly, when $c(x-\alpha)=$ constant, we have $\omega(x)=0$, the inequality (2.10) is reduced to the inequality (2.5).

3. Applications

As applications, in this section we will give some extensions and refinements of Widder's theorem and Hardy-Littlewood's theorem.

Let $a_n\ge 0(n=0,1,2,\ldots)$, $A(x)=\sum_{n=0}^\infty a_nx^n$, $A^*(x)=\sum_{n=0}^\infty \frac{a_nx^n}{n!}$. Then

$$\int_0^1 A^2(x)\mathrm{d}x\le \pi\int_0^\infty (e^{-x}A^*(x))^2\mathrm{d}x. \tag{3.13}$$

This is Widder's theorem (see [7]).

We shall give an extension and a refinement of (3.13) as following.

Theorem 3.3. *With the assumptions as the above-mentioned, if* $f(x)=e^{-(x-\alpha)}A^*(x-\alpha)$, *then*

$$\Big(\int_0^1 A^2(x)\mathrm{d}x\Big)^2\le\pi^2\left\{\Big(\int_0^\infty f^2(x)\mathrm{d}x\Big)^2-\Big(\int_0^\infty \omega(x)f^2(x)\mathrm{d}x\Big)^2\right\}, \tag{3.14}$$

where $\omega(x)$ *is defined by* (2.11).

Proof. At first we have the following relation:

$$\int_0^\infty e^{-t}A^*(tx)\mathrm{d}t=\int_0^\infty e^{-t}\sum_{n=0}^\infty\frac{a_n(xt)^n}{n!}\mathrm{d}t=\sum_{n=0}^\infty\frac{a_nx^n}{n!}\int_0^\infty t^ne^{-t}\mathrm{d}t$$
$$=\sum_{n=0}^\infty a_nx^n=A(x).$$

Let $tx=s-\alpha$. Then we have

$$\int_0^1 A^2(x)\mathrm{d}x=\int_0^1\left\{\int_0^\infty e^{-t}A^*(tx)\mathrm{d}t\right\}^2\mathrm{d}x=\int_0^1\Big(\int_\alpha^\infty e^{-\frac{s-\alpha}{x}}A^*(s-\alpha)\mathrm{d}s\Big)^2\frac{1}{x^2}\mathrm{d}x$$
$$=\int_1^\infty\Big(\int_\alpha^\infty e^{-(s-\alpha)y}A^*(s-\alpha)\mathrm{d}s\Big)^2\mathrm{d}y$$

$$= \int_0^\infty \Big(\int_\alpha^\infty e^{-(s-\alpha)u-(s-\alpha)} A^*(s-\alpha)\mathrm{d}s\Big)^2 \mathrm{d}u$$

$$= \int_0^\infty \Big(\int_\alpha^\infty e^{-(s-\alpha)u} f(s)\mathrm{d}s\Big)^2 \mathrm{d}u = \int_\alpha^\infty\int_\alpha^\infty \frac{f(s)f(t)}{s+t-2\alpha}\mathrm{d}s\mathrm{d}t, \tag{3.15}$$

where $f(x) = e^{-(x-\alpha)}A^*(x-\alpha)$. Using Corollary 2.1, the inequality (3.14) follows from (3.15) at once. □

Let $f(x) \in L^2(0,\ 1)$. If $a_n = \int_0^1 x^n f(x)\mathrm{d}x$, $n = 0,\ 1,\ 2, \cdots$, then we have the Hardy- Littlewood's inequality (see [4]) of the form

$$\sum_{n=0}^\infty a_n^2 \le \pi \int_0^1 f^2(x)\mathrm{d}x, \tag{3.16}$$

where π is the best constant that keeps (3.16) valid. In our previous paper [1], the inequality (3.16) was extended and established the following inequality:

$$\int_0^\infty f^2(x)\mathrm{d}x \le \pi \int_0^1 h^2(x)\mathrm{d}x, \tag{3.17}$$

where $f(x) = \int_0^1 t^x h(x)\mathrm{d}x\ x \in [0, +\infty)$.

The inequality (3.17) is called the Hardy-Littlewood integral inequality. Afterwards the inequality (3.17) was refined into the following form (see [3]):

$$\int_0^\infty f^2(x)\mathrm{d}x \le \pi \int_0^1 t\ h^2(t)\mathrm{d}t. \tag{3.18}$$

We will extend and further refine the inequality (3.18) here.

Theorem 3.4. *Let $0 < \mu \le 2$, $0 < \frac{\mu}{\lambda} < 2$, $h(t) \in L^2(0,\ 1)$ and $h(t) \ne 0$. Define a function by $f(x) = \int_0^1 t^{(x-\alpha)^\lambda} |h(t)|\,\mathrm{d}t$. If $0 < \int_\alpha^{+\infty} (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x < +\infty$, then*

$$\Big(\int_\alpha^\infty f^2(x)\mathrm{d}x\Big)^4 \le \Big(\frac{\pi}{\lambda\sin\left(\frac{\mu\pi}{2\lambda}\right)}\Big)^2 \left\{\Big(\int_\alpha^\infty (x-\alpha)^{1-\lambda} f^2(x)\mathrm{d}x\Big)^2 - \Big(\int_\alpha^\infty \tilde{\omega}(x) f^2(x)\mathrm{d}x\Big)^2\right\} \Big(\int_0^1 t\ h^2(t)\mathrm{d}t\Big)^2, \tag{3.19}$$

where the weight function $\tilde{\omega}(x)$ is defined by (2.6).

Proof. Let us write $f^2(x)$ in form:

$$f^2(x) = \int_0^1 f(x) t^{(x-\alpha)^\lambda} |h(t)|\,\mathrm{d}t.$$

Applying Schwarz's inequality, we have

$$\Big(\int_\alpha^{+\infty} f^2(x)\mathrm{d}x\Big)^4 = \left\{\int_\alpha^\infty \Big(\int_0^1 f(x) t^{(x-\alpha)^\lambda} |h(t)|\,\mathrm{d}t\Big)\mathrm{d}x\right\}^4$$

$$= \left\{\int_0^1 \Big(\int_\alpha^{+\infty} f(x) t^{(x-\alpha)^\lambda - \frac{1}{2}}\mathrm{d}x\Big) t^{\frac{1}{2}} |h(t)|\,\mathrm{d}t\right\}^4$$

$$\leq \left\{ \int_0^1 \Big(\int_\alpha^{+\infty} f(x) t^{(x-\alpha)^\lambda - \frac{1}{2}} \mathrm{d}x \Big)^2 \mathrm{dt} \int_0^1 \mathrm{t\ h^2(t)dt} \right\}^2$$
$$= \left\{ \int_0^1 \Big(\int_\alpha^{+\infty} \int_\alpha^{+\infty} f(x) f(y) t^{(x-\alpha)^\lambda + (y-\alpha)^\lambda - 1} \mathrm{d}x\mathrm{d}y \Big) \mathrm{dt} \int_0^1 \mathrm{t\ h^2(t)dt} \right\}^2$$
$$= \Big(\int_\alpha^{+\infty} \int_\alpha^{+\infty} \frac{f(x)f(y)}{(x-\alpha)^\lambda + (y-\alpha)^\lambda} \mathrm{d}x\mathrm{d}y \Big)^2 \Big(\int_0^1 t\, h^2(t) \mathrm{dt} \Big)^2. \tag{3.20}$$

By (2.8), the inequality (3.19) follows from (3.20) at once. □

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