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On the weak compactness of the product of some operators

BELMESNAOUI AQZZOUZ, OTHMAN ABOUTAFAIL and AZIZ ELBOUR

ABSTRACT. We study the weak compactness of the product of two operators when one of them is order weakly compact (resp. semi-compact, b-weakly compact).

1. PRELIMINARIES

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. It is well known that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For more information on positive operators, we refer the reader to [1].

An operator T from a Banach lattice E into a Banach space X is called order weakly compact whenever $T[-x, x]$ is relatively weakly compact for every $x \in E^+$ where $E^+ = \{x \in E : 0 \leq x\}$.

Now, an operator T from a Banach space E into a Banach lattice F is called semi-compact if for each $\varepsilon > 0$, there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where B_H is the closed unit ball of $H = E$ or F.

Contrary to weakly compact operators [1], the class of order weakly compact (resp. semi-compact) operators satisfies the domination problem i.e. if E and F are two Banach lattices and S and T are two operators from E into F such that $0 \leq S \leq T$ and T is order weakly compact (resp. semi-compact), then S is order weakly compact (resp. semi-compact).

2. MAIN RESULTS

In [1] (Theorem 5.33), it is proved that if E, F and G are Banach lattices and $E \overset{S_1}{\to} F \overset{S_2}{\to} G$ is a schema of positive operators such that S_1 and S_2 are dominated by positive weakly compact operators, then $S_2 \circ S_1$ is weakly compact.

In the following, we give a generalization of this result.

Theorem 2.1. *Let E and F be Banach lattices and let X be a Banach space. Consider the operators $S_1 : E \to F$ and $S_2 : F \to X$. If S_1 is positive and dominated by a positive weakly compact operator and S_2 is order weakly compact, then $S_2 \circ S_1$ is a weakly compact operator.*

Proof. Let $T_1 : E \to F$ be a positive weakly compact operator satisfying $0 \leq S_1 \leq T_1$ and let A be the order ideal generated by F in the topological bidual F''. By Theorem 5.23 of

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Aliprantis-Burkinshaw [1], the weak compactness of T_1 implies that $T_1(E'') \subseteq F$. Thus, if $0 \le x'' \in E''$, then (in view of $0 \le S_1'' \le T_1''$) we have $0 \le S_1''(x'') \le T_1''(x'') \in F$, and so

$$S_1''(E'') \subseteq A.$$

On the other hand, since $S_2 : F \to X$ is order weakly compact then, by Theorem 5.57 of Aliprantis-Burkinshaw [1], we have

$$S_2''(A) \subseteq X.$$

Then we see that

$$(S_2 \circ S_1)''(E'') = S_2''(S_1''(E'')) \subseteq S_2''(A) \subseteq X,$$

which means, by Theorem 5.23 of Aliprantis-Burkinshaw [1], that $S_2 \circ S_1$ is a weakly compact operator. This end the proof. □

As a direct consequence of Theorem 2.1, we obtain Theorem 5.32 of Aliprantis-Burkinshaw [1] on the domination for weakly compact operators.

Corollary 2.1. *If a positive operator on a Banach lattice is dominated by a weakly compact operator, then its square is weakly compact.*

Let us recall that a norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|.\|$ where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

There exist Banach lattices E and F and there exists a semi-compact and order weakly compact operator $T : E \longrightarrow F$ which is not weakly compact. In fact, consider the operators $S, T : E \to F$ constructed in the Example 5.30 of Aliprantis-Burkinshaw [1] with T has rank one, S not weakly compact and $0 \le S \le T$. It is clear that T is compact (because T has rank one). Hence T is semi-compact and order weakly compact. Now, it follows from Theorem 5.72 and Corollary 5.53 of Aliprantis-Burkinshaw [1] that S is also semi-compact and order weakly compact. But S is not weakly compact.

More general, if neither E' nor F has an order continuous norm, then it follows from the proof of the implication (2)⟹(1) of Theorem 5.31 of Aliprantis-Burkinshaw [1] that there exists two operators $S, T : E \to F$ such that T has rank one, S not weakly compact and $0 \le S \le T$. It is clear that S is semi-compact and order weakly compact which is not weakly compact.

However, we have

Theorem 2.2. *Let X and Y be two Banach spaces and let E be a Banach lattice. If $T : X \to E$ is a semi-compact operator and $S : E \to Y$ is an order weakly compact operator, then $S \circ T$ is weakly compact.*

Proof. Let $T : X \to E$ be a semi-compact operator and $S : E \to Y$ an order weakly compact operator, and let $\varepsilon > 0$. Since T is semi-compact, there exists some $u \in E^+$ such that $T(B_X) \subset [-u, u] + \varepsilon B_E$. Hence

$$(S \circ T)(B_X) \subset S([-u, u] + \varepsilon B_E) \subset S[-u, u] + \varepsilon S(B_E) \subset S([-u, u]) + \varepsilon \|S\| B_Y.$$

Then we have

$$(S \circ T)(B_X) \subset S([-u, u]) + \varepsilon \|S\| B_Y.$$

On the other hand, the order weak compactness of S implies that $S([-u, u])$ is weakly relatively compact. So, by Theorem 3.44 of Aliprantis-Burkinshaw [1], the subset $(S \circ T)(B_X)$ is weakly relatively compact. This proves that $S \circ T$ is weakly compact. □

As a consequence of Theorem 2.2, we obtain that the second power of a semi-compact and order weakly compact operator is always weakly compact. In fact,

Corollary 2.2. *Let E be a Banach lattice, then for each semi-compact and order weakly compact operator T from E into E the second power operator T^2 is weakly compact.*

Now, a Banach lattice E is said to be a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. For an example, each reflexive Banach lattice (resp. AL-space) is a KB-space.

Note that each KB-space has an order continuous norm, but a Banach lattice with an order continuous norm is not necessary a KB-space. In fact, the Banach lattice c_0 has an order continuous norm but it is not a KB-space. However, if E is a Banach lattice, the topological dual E' is a KB-space if and only if its norm is order continuous.

Recall from [2] that a subset A of a Banach lattice E is called b-order bounded in E if it is order bounded in the topological bidual E''.

An operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact whenever T carries each b-order bounded subset of E into a relatively weakly compact subset of X. It is clear that each weakly compact operator is b-weakly compact but the converse is false in general. Also, each b-weakly compact operator is order weakly compact, but the converse is not true in general.

The following result gives a generalization of Proposition 7 (3) of [3].

Proposition 2.1. *Let E and F be Banach lattices, X a Banach space and let $T : E \to F$, $S : F \to X$ be two operators. If F has an order continuous norm and S is b-weakly compact, then $S \circ T$is a b-weakly compact operator.*

Proof. Since $S : F \to X$ is b-weakly compact and F has an order continuous norm, it follows from Proposition 2 of [3] that there exist a KB-space G, an interval preserving lattice homomorphism $Q : F \to G$ and an operator $R : G \to X$ such that $S = R \circ Q$. Hence $S \circ T$ factors through G which is a KB-space. So, by Corollary 2.4 of [4], the operator $S \circ T$ is b-weakly compact. □

By combining Proposition 2 of Altin [3] and Theorem 5.27 of [1], we obtain

Proposition 2.2. *Let E and F be Banach lattices, X a Banach space, and let $E \xrightarrow{T} F \xrightarrow{S} X$ be operators. If E' and F have order continuous norms and S is b-weakly compact, then $S \circ T$ is weakly compact.*

Proof. By Proposition 2 of Altin [3], there exist a KB-space G, an interval preserving lattice homomorphism $Q : F \to G$ and an operator $R : G \to X$ such that $S = R \circ Q$. Since E' has an order continuous norm and G is a KB-space, it follows from Theorem 5.27 of [1] that $Q \circ T : E \to G$ is weakly compact, and so $S \circ T = R \circ (Q \circ T)$ is weakly compact. □

As a consequence of Proposition 6, we obtain.

Corollary 2.3. *Let E be a Banach lattice, X a Banach space and $T : E \longrightarrow X$ an operator. If E and E' have order continuous norms and T is b-weakly compact, then T is weakly compact.*

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DÉPARTEMENT D'ECONOMIE
UNIVERSITÉ MOHAMMED V-SOUISSI
FACULTÉ DES SCIENCES ECONOMIQUES, JURIDIQUES ET SOCIALES
B.P. 5295, SALAALJADIDA, MOROCCO
E-mail address: baqzzouz@hotmail.com

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ IBN TOFAIL
FACULTÉ DES SCIENCES
B.P. 133, KÉNITRA, MOROCCO