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A new generalization of Radon's Inequality and applications

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ABSTRACT. In this paper we prove a new generalization of Radon's Inequality and give some applications.

1. INTRODUCTION

Let $\mathbb{N}$ be positive integers, $\mathbb{N} = \{1, 2, \dots\}$. The inequality from (1.1) is called, in literature, Bergström's Inequality (see [1], [5]–[10], [12]–[15] or [18]).

Theorem 1.1. *If $n \in \mathbb{N}$, $x_k \in \mathbb{R}$ and $y_k > 0$, $k \in \{1, 2, \dots, n\}$, then*

$$\frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + x_2 + \cdots + x_n)^2}{y_1 + y_2 + \cdots + y_n}, \tag{1.1}$$

with equality if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2} = \cdots = \dfrac{x_n}{y_n}$.

In [19], J. Radon proves the inequality from Theorem 1.2, later called Radon's Inequality. Several particular or similar variants of this inequality have been published in the following Romanian journals: Gazeta Matematică, Recreaţii Matematice issued by the team from Iaşi and Revista de Matematică din Timişoara (see [3], [4], [11] and [17]).

Theorem 1.2. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \dots, n\}$ and $m \geq 0$, then*

$$\frac{x_1^{m+1}}{y_1^m} + \frac{x_2^{m+1}}{y_2^m} + \cdots + \frac{x_n^{m+1}}{y_n^m} \geq \frac{(x_1 + x_2 + \cdots + x_n)^{m+1}}{(y_1 + y_2 + \cdots + y_n)^m}. \tag{1.2}$$

In [2], we prove the inequality from Theorem 1.3, which is a generalization of Radon's Inequality.

Theorem 1.3. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \dots, n\}$, $m \geq 0$ and $p \geq 1$, then*

$$\frac{x_1^{m+p}}{y_1^m} + \frac{x_2^{m+p}}{y_2^m} + \cdots + \frac{x_n^{m+p}}{y_n^m} \geq \frac{(x_1 y_1^{p-1} + x_2 y_2^{p-1} + \cdots + x_n y_n^{p-1})^{m+p}}{(y_1^p + y_2^p + \cdots + y_n^p)^{m+p-1}}, \tag{1.3}$$

with equality if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2} = \cdots = \dfrac{x_n}{y_n}$.

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2. Main results and applications

In this section, we give a new generalization of Radon's Inequality and some applications. The result of the Theorem 2.1 is included in [12]. In this paper, we present a new and complete proof of this result.

Theorem 2.4. *If $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$ and $m \geq p \geq 0$, then*

$$\frac{x_1^{m+1}}{y_1^p} + \frac{x_2^{m+1}}{y_2^p} + \cdots + \frac{x_n^{m+1}}{y_n^p} \geq n^{p-m} \frac{(x_1 + x_2 + \cdots + x_n)^{m+1}}{(y_1 + y_2 + \cdots + y_n)^p}, \tag{2.4}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$.

Proof. In the following, we denote $X_n = x_1 + x_2 + \cdots + x_n$ and $Y_n = y_1 + y_2 + \cdots + y_n$. The left side from (2.4) can be written as

$$\sum_{k=1}^{n} \frac{x_k^{m+1}}{y_k^p} = Y_n \sum_{k=1}^{n} \frac{y_k}{Y_n} t_k^{p+1}, \tag{2.5}$$

where $t_k = \dfrac{x_k^{\frac{m+1}{p+1}}}{y_k}$, $k \in \{1, 2, \ldots, n\}$.

The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(t) = t^{p+1}$, $t \in (0, \infty)$, is convex on $(0, \infty)$, so we have that: $\sum\limits_{k=1}^{n} \dfrac{y_k}{Y_n} t_k^{p+1} \geq \left(\sum\limits_{k=1}^{n} \dfrac{y_k}{Y_n} t_k \right)^{p+1}$, from where

$$\sum_{k=1}^{n} \frac{y_k}{Y_n} t_k^{p+1} \geq \frac{1}{Y_n^{p+1}} \left(\sum_{k=1}^{n} x_k^{\frac{m+1}{p+1}} \right)^{p+1}. \tag{2.6}$$

But, the function $g : (0, \infty) \to \mathbb{R}$ defined by $g(x) = x^{\frac{m+1}{p+1}}$, $x \in (0, \infty)$, is convex on $(0, \infty)$ and then

$$\sum_{k=1}^{n} g(x_k) \geq n g \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right),$$

from where

$$\sum_{k=1}^{n} x_k^{\frac{m+1}{p+1}} \geq n^{\frac{p-m}{p+1}} X_n^{\frac{m+1}{p+1}}. \tag{2.7}$$

From (2.5)-(2.7), inequality (2.4) follows. □

Remark 2.1. If we consider $m = p$ in inequality (2.4), we obtain Radon's Inequality.

Application 2.1. If $n \in \mathbb{N}$, $x_k \geq 0$, $a, b \in \mathbb{R}$, $a(x_1 + x_2 + \cdots + x_n) - bx_k > 0$ for any $k \in \{1, 2, \ldots, n\}$ and $m \geq p \geq 0$, then

$$\sum_{k=1}^{n} \frac{x_k^{m-p+1}}{(a(x_1+x_2+\cdots+x_n)-bx_k)^p} \geq n^{2p-m} \frac{(x_1+x_2+\cdots+x_n)^{m-2p+1}}{(an-b)^p}. \tag{2.8}$$

Solution. By using inequality (2.4), we have that

$$\sum_{k=1}^{n} \frac{x_k^{m-p+1}}{(a(x_1+x_2+\cdots+x_n)-bx_k)^p} = \sum_{k=1}^{n} \frac{x_k^{m+1}}{(ax_k X_n - bx_k^2)^p} \geq n^{p-m} \frac{X_n^{m+1}}{\left(aX_n \sum\limits_{k=1}^{n} x_k - b \sum\limits_{k=1}^{n} x_k^2\right)^p},$$

so

$$\sum_{k=1}^{n} \frac{x_k^{m-p+1}}{(a(x_1+x_2+\cdots+x_n)-bx_k)^p} \geq n^{p-m} \frac{X_n^{m+1}}{\left(aX_n^2 - b \sum\limits_{k=1}^{n} x_k^2\right)^p}, \tag{2.9}$$

where $X_n = x_1 + x_2 + \cdots + x_n$.

By taking the inequality $\sum\limits_{k=1}^{n} x_k^2 \geq \frac{(x_1+x_2+\cdots+x_n)^2}{n}$ into account, we have that $aX_n^2 - b\sum\limits_{k=1}^{n} x_k^2 \leq aX_n^2 - \frac{b}{n}X_n^2 = \frac{an-b}{n}X_n^2$, and then

$$\frac{X_n^{m+1}}{\left(aX_n^2 - b \sum\limits_{k=1}^{n} x_k^2\right)^p} \geq n^p \frac{X_n^{m-2p+1}}{(an-b)^p}. \tag{2.10}$$

From (2.9) and (2.10), inequality (2.8) results.

Remark 2.2. In conditions of Application 2.1, for $m = p = 1$ we obtain the inequality

$$\sum_{k=1}^{n} \frac{x_k}{aX_n - bx_k} \geq \frac{n}{an-b}. \tag{2.11}$$

The inequality from (2.11) is a generalization of Nesbitt's Inequality:

$$\frac{x_1}{x_2+x_3} + \frac{x_2}{x_3+x_1} + \frac{x_3}{x_1+x_2} \geq \frac{3}{2}, \tag{2.12}$$

for any $x_1, x_2, x_3 > 0$ (see [16]).

Application 2.2. If $a, b, c > 0$, prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{(ab+bc+ca)^3}{6a^3b^3c^3(a+b+c)} \geq \frac{ab+bc+ca}{2a^2b^2c^2} \geq \frac{3}{2} \cdot \frac{1}{\sqrt[3]{(abc)^4}}. \tag{2.13}$$

Solution. If $n = 3$, $m = 2$, $p = 1$, $x_1 = \frac{1}{a}$, $x_2 = \frac{1}{b}$, $x_3 = \frac{1}{c}$, $y_1 = b + c$, $y_2 = c + a$, $y_3 = a + b$, $y_3 = a + b$, from (2.4) the first inequality above follows. By using inequality $(x+y+z)^2 \geq 3(xy+yz+zx)$, for any $x, y, z \in \mathbb{R}$, we have that $(ab+bc+ca)^2 \geq 3abc(a+b+c)$, so inequality $\frac{(ab+bc+ca)^3}{6a^3b^3c^3(a+b+c)} \geq \frac{ab+bc+ca}{2a^2b^2c^2}$ holds.

But $ab + bc + ca \geq 3\sqrt[3]{a^2b^2c^2}$ and then $\frac{ab+bc+ca}{2a^2b^2c^2} \geq \frac{3}{2} \cdot \frac{1}{\sqrt[3]{(abc)^4}}$.

From the inequalities above, (2.13) follows.

Remark 2.3. From this application, we obtain the following problem given in the $XXXVI^{th}$ International Mathematical Olympiad, Canada, 1995:

If $a, b, c > 0$ with $abc = 1$, then

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}. \tag{2.14}$$

By using inequality (2.4), we can prove the following inequality:

If $n \in \mathbb{N}$, $a, b, x_1, x_2, \ldots, x_n > 0$, $m \geq p \geq 0$, $x_1 \cdot x_2 \cdot \ldots \cdot x_n = 1$ and by definition $x_{n+1} = x_1$, then the inequality

$$\sum_{k=1}^{n} \frac{x_k^p}{(ax_{k+1} + bx_k)^p \cdot x_{k+1}^{m-p+1}} \geq \frac{n}{(a+b)^p} \tag{2.15}$$

holds.

Remark 2.4. If $n = 3$, $a = b = p = 1$ and $m = 4$, from (2.15) inequality (2.14) results, so the inequality from (2.15) is a generalization for the inequality from (2.14).

Application 2.3. If $a, b, c, d, x, y, z > 0$, $m \geq p \geq 0$ and $b + c + d \leq 24a^2$, then

$$\frac{x^{m-p+1}}{\sqrt{(a^2x^2 + byz)^p}} + \frac{y^{m-p+1}}{\sqrt{(a^2y^2 + czx)^p}} + \frac{z^{m-p+1}}{\sqrt{(a^2z^2 + dxy)^p}} \geq \frac{(x+y+z)^{m-2p+1}}{a^p}. \tag{2.16}$$

Solution. We note with A the left member of the inequality from (2.16). We have that

$$A = \sum \frac{x^{m+1}}{x^p\sqrt{(a^2x^2 + byz)^p}} = \sum \frac{x^{m+1}}{\left(\sqrt{x}\sqrt{a^2x^3 + bxyz}\right)^p}$$

and by using inequality (2.4), we obtain

$$A \geq 3^{p-m} \frac{(x+y+z)^{m+1}}{\left(\sqrt{x}\sqrt{a^2x^3+bxyz} + \sqrt{y}\sqrt{a^2y^3+cxyz} + \sqrt{z}\sqrt{a^2z^3+dxyz}\right)^p}. \tag{2.17}$$

If we note $B = \sqrt{x}\sqrt{a^2x^3+bxyz} + \sqrt{y}\sqrt{a^2y^3+cxyz} + \sqrt{z}\sqrt{a^2z^3+dxyz}$, by using Cauchy-Schwarz Inequality, we have

$$B^2 \leq (x+y+z)(a^2(x^3+y^3+z^3) + (b+c+d)xyz)$$

and by taking the relation $b + c + d \leq 24a^2$ into account, it results that

$$B^2 \leq a^2(x+y+z)(x^3+y^3+z^3+24xyz).$$

But the inequality $x^3 + y^3 + z^3 + 24xyz \leq (x+y+z)^3$, $x, y, z > 0$, is well known, and then the inequality above becomes

$$B \leq a(x+y+z)^2. \tag{2.18}$$

From (2.17) and (2.18), inequality (2.16) follows.

Remark 2.5. From inequality (2.16), for $a = 1$, $b = c = d = 8$, $m = p = 1$, we obtain the following problem given in the $XLII^{th}$ International Mathematical Olympiad, USA, 2001:

If $x, y, z > 0$, then inequality

$$\frac{x}{\sqrt{x^2+8yz}} + \frac{y}{\sqrt{y^2+8zx}} + \frac{z}{\sqrt{z^2+8xy}} \geq 1 \tag{2.19}$$

holds.

Application 2.4. If $n \in \mathbb{N}$, $(F_n)_{n\geq 0}$ is Fibonacci's sequence, $a > \dfrac{F_n}{F_{n+2}-1}$ and $m \geq p \geq 0$, then

$$\sum_{k=1}^{n} \frac{F_k^{m+1}}{(a(F_{n+2}-1)-F_k)^p} \geq \frac{n^{p-m}}{(an-1)^p}\,(F_{n+2}-1)^{m-p+1}. \tag{2.20}$$

Solution. It is well-known that $\sum\limits_{k=1}^{n} F_k = F_{n+2}-1$. By using inequality (2.4), we have that

$$\sum_{k=1}^{n} \frac{F_k^{m+1}}{(a(F_{n+2}-1)-F_k)^p} \geq n^{p-m} \frac{\left(\sum\limits_{k=1}^{n} F_k\right)^{m+1}}{\left(na(F_{n+2}-1)-\sum\limits_{k=1}^{n} F_k\right)^p},$$

from where the inequality from (2.20) follows.

Theorem 2.5. *If $a,b \in \mathbb{R}$, $a<b$, $m \geq p \geq 0$, $f,g:[a,b] \to [0,\infty)$ are integrable function on $[a,b]$, $g(x) \neq 0$ for any $x \in [a,b]$, then*

$$\int_a^b \frac{(f(x))^{m+1}}{(g(x))^p}\,dx \geq (b-a)^{p-m} \frac{\left(\int_a^b f(x)dx\right)^{m+1}}{\left(\int_a^b g(x)dx\right)^p}. \tag{2.21}$$

Proof. Let $n \in \mathbb{N}$ and $x_k = a + k\dfrac{b-a}{n}$, $k \in \{0,1,\ldots,n\}$. By Theorem 2.1, we get that

$$\sum_{k=1}^{n} \frac{(f(x_k))^{m+1}}{(g(x_k))^p} \geq n^{p-m} \frac{\left(\sum\limits_{k=1}^{n} f(x_k)\right)^{m+1}}{\left(\sum\limits_{k=1}^{n} g(x_k)\right)^p}.$$

It results that

$$\sigma\left(\frac{f^{m+1}}{g^p},\Delta_n,x_k\right) \geq (b-a)^{p-m} \frac{(\sigma(f,\Delta_n,x_k))^{m+1}}{(\sigma(g,\Delta_n,x_k))^P},$$

where $\sigma\left(\dfrac{f^{m+1}}{g^p},\Delta_n,x_k\right)$ is the corresponding Riemann sum of function $\dfrac{f^{m+1}}{g^p}$, of $\Delta_n = (x_0,x_1,\ldots,x_n)$ division, and the intermediate x_k points. By passing to limit in the inequality above, when n tends to infinity, the inequality (2.21) follows. □

Remark 2.6. By particularization, from (2.21) we can obtain the classic integral inequalities: Hölder's Inequality, Cauchy-Schwarz's Inequality and Chebyshev's Inequality.

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