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When is the limit equal to the supremum norm of f?

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ABSTRACT. If f is a nonnegative continuous function on $[0,1]$ we investigate the problem when is $\lim_{n\to\infty}\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx}$ equal to the supremum norm of f. This problem is motivated by a problem in classical analysis which states that if f is a continuous function on $[a,b]$ then the following equality holds $\lim_{n\to\infty}\sqrt[n]{\int_a^b |f(x)|^n dx} = \|f\|_\infty$.

1. INTRODUCTION AND THE MAIN RESULT

It is a problem in classical analysis to show, (see [1]), that if f is a continuous function on $[a,b]$ then $\lim_{n\to\infty}\sqrt[n]{\int_a^b |f(x)|^n dx} = \|f\|_\infty$. Motivated by this problem we let $f:[0,1]\to[0,\infty)$ be a continuous function and we investigate the problem when is

$$\lim_{n\to\infty}\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} = \|f\|_\infty. \tag{1.1}$$

We prove that equality holds in (1.1) provided that f attains its maximum at 0 or 1, and for the contrary case we give an example where equality (1.1) fails to hold. Our main result is the following theorem.

Theorem 1.1. *Let $f:[0,1]\to[0,\infty)$ be a continuous function that attains its maximum either at 0 or at 1. Then*

$$\lim_{n\to\infty}\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} = \|f\|_\infty.$$

Remark 1.1. It is interesting to study whether equality (1.1) still holds whenever f is a function that does not attain its maximum at 0 or 1. We conjecture that, in this case, strict inequality holds in (1.1) and we give below an example in favor of this conjecture.

2. PROOF OF THE MAIN RESULT

Proof. Let $M=\|f\|_\infty$. If $M=0$ the equality to prove follows by triviality so we consider the case when $M>0$. We have $\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx}\le M$. Thus, it suffices to prove that

$$\varliminf_{n\to\infty}\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx}\ge M.$$

First we consider the case when f attains its maximum at 1, i.e., $M=f(1)$. Let $0<\epsilon<M$. Using the continuity of f at 1 we get that there is $\delta=\delta(\epsilon)>0$ such that $M-\epsilon\le f(x)\le M$

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for all $\delta \le x \le 1$. Since the functions $x \to f(x^k)$, $k = 1, \ldots, n$, also attain their maximum at 1 we have that $M - \epsilon \le f(x^k) \le M$ for $\sqrt[k]{\delta} \le x \le 1$. On the other hand, since $\delta < 1$ and $\delta < \sqrt{\delta} < \cdots < \sqrt[n]{\delta}$, we have that for all $k = 1, \ldots, n$, one has $M - \epsilon \le f(x^k) \le M$, for $\sqrt[n]{\delta} \le x \le 1$. Thus,

$$\int_0^1 f(x)f(x^2)\cdots f(x^n)dx \ge \int_{\sqrt[n]{\delta}}^1 f(x)f(x^2)\cdots f(x^n)dx \ge (M-\epsilon)^n \left(1 - \sqrt[n]{\delta}\right),$$

and it follows that

$$\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} \ge (M-\epsilon)\ \sqrt[n]{1 - \sqrt[n]{\delta}}.$$

Using that $\lim\limits_{n\to\infty} \sqrt[n]{1 - \sqrt[n]{\delta}} = 1$, we get that

$$\varliminf_{n\to\infty} \sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} \ge M - \epsilon,$$

and since ϵ was arbitrary taken the result follows.

Now we consider the case when f attains its maximum at 0, i.e., $M = f(0)$. Let $0 < \epsilon < f(0)$ be fixed. Using the continuity of f at 0 we get that there is $\delta > 0$ such that $0 < f(0) - \epsilon < f(x) < f(0)$ for all $0 < x < \delta$. Since $x^k < x$ for $k \in \mathbb{N}$ and $x \in (0, \delta)$ one has that $f(x^k) > f(0) - \epsilon > 0$. We have

$$\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} \ge \sqrt[n]{\int_0^\delta f(x)f(x^2)\cdots f(x^n)dx}. \tag{2.2}$$

It follows, based on Bernoulli's integral inequality ([2, Corolar 4, p. 8]), that

$$\begin{aligned} \sqrt[n]{\int_0^\delta f(x)f(x^2)\cdots f(x^n)dx} &\ge \left(\int_0^\delta dx\right)^{\frac{1}{n}-1} \cdot \int_0^\delta \sqrt[n]{f(x)f(x^2)\cdots f(x^n)}dx \\ &= \delta^{\frac{1}{n}-1} \int_0^\delta \sqrt[n]{f(x)f(x^2)\cdots f(x^n)}dx. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) we obtain that

$$\sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} \ge \delta^{\frac{1}{n}-1} \cdot \int_0^\delta \sqrt[n]{f(x)f(x^2)\cdots f(x^n)}dx. \tag{2.4}$$

We prove that

$$\lim_{n\to\infty} \int_0^\delta \sqrt[n]{f(x)f(x^2)\cdots f(x^n)}dx = \delta \cdot f(0) \tag{2.5}$$

Let

$$h_n(x) = \sqrt[n]{f(x)f(x^2)\cdots f(x^n)}, \quad x \in (0, \delta),$$

and let v be the constant function $v(x) = M = f(0)$. Then $h_n(x) \le v(x)$ for all $x \in (0, \delta)$. On the other hand, $\ln h_n(x) = \dfrac{1}{n}\sum\limits_{k=1}^{n} \ln f(x^k)$, and note that $\ln$ is well defined since $f(x^k) > 0$ for $x \in (0, \delta)$. It follows, based on Cesaro Stolz Lemma, ([4, Glossary, p. 435]), that

$$\lim_{n\to\infty} \ln h_n(x) = \lim_{n\to\infty} \ln f(x^{n+1}) = \ln f(0).$$

Thus, $\lim_{n\to\infty} h_n(x) = f(0)$ and equality (2.5) follows based on Lebesgue Convergence Theorem ([3, Theorem 16, p. 91]). Combining (2.4) and (2.5) we obtain that

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{\int_0^1 f(x)f(x^2)\cdots f(x^n)dx} &\geq \lim_{n\to\infty}\left(\delta^{\frac{1}{n}-1}\cdot\int_0^\delta \sqrt[n]{f(x)f(x^2)\cdots f(x^n)}dx\right)\\ &= f(0),\end{aligned}$$

and the theorem is proved. $\square$

Now we give an example where f does not attain its maximum at 0 or 1 and equality (1.1) fails to hold. Let $f : [0,1] \to [0,1]$ be the continuous function defined by

$$f(x) = \begin{cases} 1 - e^{-\frac{4}{(2x-1)^2}}, & x \neq \dfrac{1}{2},\\ 1, & x = \dfrac{1}{2},\end{cases}$$

and let L be the value of the limit

$$L = \lim_{n\to\infty}\sqrt[n]{\begin{array}{l}\displaystyle\int_0^{1/2}(1-e^{-\frac{4}{(2x-1)^2}})\cdots(1-e^{-\frac{4}{(2x^n-1)^2}})dx\\ \displaystyle +\int_{1/2}^1(1-e^{-\frac{4}{(2x-1)^2}})\cdots(1-e^{-\frac{4}{(2x^n-1)^2}})dx\end{array}}.$$

We note that f increases on $[0,1/2]$ and that $x^n < x^{n-1} < \cdots < x^2$, from which it follows that $f(x)f(x^2)\cdots f(x^n) < f(x^2)^{n-1} < (f(1/4))^{n-1} = (1-e^{-16})^{n-1}$. We have, since $1 - e^{-4/(2x-1)^2} < 1$, that

$$\int_0^{1/2}(1-e^{-\frac{4}{(2x-1)^2}})(1-e^{-\frac{4}{(2x^2-1)^2}})\cdots(1-e^{-\frac{4}{(2x^n-1)^2}})dx \leq \frac{1}{2}(1-e^{-16})^{n-1},$$

and hence L is less than or equal to

$$\lim_{n\to\infty}\sqrt[n]{\begin{array}{c}\displaystyle\tfrac{1}{2}(1-e^{-16})^{n-1} + \sum_{k=1}^{n-1}\int_{(\frac{1}{2})^{\frac{1}{k}}}^{(\frac{1}{2})^{\frac{1}{k+1}}}(1-e^{-\frac{4}{(2x-1)^2}})\cdots(1-e^{-\frac{4}{(2x^n-1)^2}})dx\\ \displaystyle +\int_{(\frac{1}{2})^{\frac{1}{n}}}^1(1-e^{-\frac{4}{(2x-1)^2}})(1-e^{-\frac{4}{(2x^2-1)^2}})\cdots(1-e^{-\frac{4}{(2x^n-1)^2}})dx\end{array}}.$$

Let $k = 1, 2, \ldots, n$, be fixed and let A be the following set

$$A = \left\{m, (2x^m-1)^2 \geq \frac{1}{2}, m = 1, 2, \ldots, n, x \in \left[(1/2)^{\frac{1}{k}}, (1/2)^{\frac{1}{k+1}}\right]\right\}.$$

We note that the set A is the set of all integers m for which the inequality, $(2x^m-1)^2 \geq \dfrac{1}{2}$, holds for x in the specified interval. We prove that the number of elements of A, i.e., the cardinality of A, verifies the inequality $|A| > n(a/b) - a - 1$, where a and b are defined bellow. Let $x \in \left[(1/2)^{\frac{1}{k}}, (1/2)^{\frac{1}{k+1}}\right]$ and let $f_m(x) = (2x^m-1)^2$. A calculation shows that

$$f_m'(x) = \begin{cases} 4mx^{m-1}(2x^m-1) > 0, & m < k,\\ 4mx^{m-1}(2x^m-1) < 0, & m > k+1.\end{cases}$$

It follows that, for $x \in \left[(1/2)^{\frac{1}{k}}, (1/2)^{\frac{1}{k+1}}\right]$, one has

$$f_m(x) \geq \begin{cases} (2^{1-\frac{m}{k}} - 1)^2, & m < k, \\ (2^{1-\frac{m}{k+1}} - 1)^2, & m > k+1. \end{cases}$$

We consider the inequalities

$$\begin{cases} (2^{1-\frac{m}{k}} - 1)^2 \geq \dfrac{1}{2}, & m < k, \\ (2^{1-\frac{m}{k+1}} - 1)^2 \geq \dfrac{1}{2}, & m > k+1, \end{cases}$$

which have the solutions

$$\begin{cases} m \leq \left(1 - \dfrac{1}{\ln 2} \ln \dfrac{\sqrt{2}+1}{\sqrt{2}}\right) k = 0.2284466968\ldots \cdot k, & m < k, \\ m \geq \left(1 - \dfrac{1}{\ln 2} \ln \dfrac{\sqrt{2}-1}{\sqrt{2}}\right) (k+1) = 2.771553303\ldots \cdot (k+1), & m > k+1. \end{cases} \tag{2.6}$$

Let $a = 1 - \dfrac{1}{\ln 2} \ln \dfrac{\sqrt{2}+1}{\sqrt{2}}$, $b = 1 - \dfrac{1}{\ln 2} \ln \dfrac{\sqrt{2}-1}{\sqrt{2}}$, and note that $\dfrac{a}{b} = 0.082425511...$ It follows, based on (2.6), that

$$m \in ([1, \lfloor ak \rfloor] \cap \mathbb{N}) \cup ([\lfloor b(k+1) \rfloor, n] \cap \mathbb{N}), \tag{2.7}$$

where $\lfloor x \rfloor$ denotes the integer part of x. We distinguish here several cases.
Case 1. $b(k+1) > n$. We have, based on (2.7), that

$$|A| = \lfloor ak \rfloor \geq ak - 1 \geq n\frac{a}{b} - a - 1.$$

Case 2. $ak < 1$. We have, based on (2.7), that

$$|A| = n - \lfloor b(k+1) \rfloor + 1 \geq n - b(k+1) \geq n - \frac{b}{a} - b > n\frac{a}{b} - a - 1.$$

Case 3. $1 < ak < b(k+1) < n$. In this case we get, based on (2.7), that

$$|A| = n - \lfloor b(k+1) \rfloor + 1 + \lfloor ak \rfloor \geq n - b(k+1) + ka \geq n\frac{a}{b} - a.$$

It follows that

$$\begin{aligned} &\int_{(\frac{1}{2})^{\frac{1}{k}}}^{(\frac{1}{2})^{\frac{1}{k+1}}} (1 - e^{-\frac{4}{(2x-1)^2}})(1 - e^{-\frac{4}{(2x^2-1)^2}}) \cdots (1 - e^{-\frac{4}{(2x^n-1)^2}}) dx \\ &= \int_{(\frac{1}{2})^{\frac{1}{k}}}^{(\frac{1}{2})^{\frac{1}{k+1}}} \prod_{m \in A} \left(1 - e^{-4/(2x^m-1)^2}\right) \cdot \prod_{m \notin A} \left(1 - e^{-4/(2x^m-1)^2}\right) dx \\ &\leq \int_{(\frac{1}{2})^{\frac{1}{k}}}^{(\frac{1}{2})^{\frac{1}{k+1}}} (1 - e^{-16})^{|A|} dx = \left(\left(\frac{1}{2}\right)^{1/(k+1)} - \left(\frac{1}{2}\right)^{1/k}\right) (1 - e^{-16})^{|A|} \\ &\leq \left(\left(\frac{1}{2}\right)^{1/(k+1)} - \left(\frac{1}{2}\right)^{1/k}\right) (1 - e^{-16})^{(n(a/b)-a-1)}. \end{aligned}$$

Thus, L is less than or equal to

$$\lim_{n\to\infty} \sqrt[n]{\frac{(1-e^{-16})^{n-1}}{2} + \left(1 - 2^{\frac{-1}{n}} + \sum_{k=1}^{n-1}\left(2^{\frac{-1}{k+1}} - 2^{\frac{-1}{k}}\right)\right)(1-e^{-16})^{(n\frac{a}{b}-a-1)}}$$
$$\leq \lim_{n\to\infty} \sqrt[n]{(1-e^{-16})^{(n(a/b)-a-1)}} = (1-e^{-16})^{a/b} = 0.9999999907242302...$$
$$< ||f||_\infty = 1,$$

Remark 2.2. One can also prove that if $g : [0,1] \to [0,\infty)$ is a continuous function that attains its maximum at 1 then

$$\lim_{n\to\infty} \sqrt[n]{\int_0^1 g(x)g(\sqrt{x})\cdots g(\sqrt[n]{x})dx} = ||g||_\infty. \tag{2.8}$$

A natural question would be to determine whether equality holds in (2.8) when g does not attain its maximum at 1. We leave this problem as an open problem to the interested reader.

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