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On γ-regular-open sets and γ-closed spaces

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ABSTRACT. The purpose of this paper is to continue studying the properties of γ-regular open sets introduced and explored by S. Hussain and B. Ahmad in 2007. The concept of γ-closed spaces have also been defined and discussed.

1. INTRODUCTION

The concept of operation γ was initiated by S. Kasahara [7]. He also introduced γ-closed graph of a function. Using this operation, H. Ogata [8] introduced the concept of γ-open sets and investigated the related topological properties of the associated topology τ_γ and τ. He further investigated general operator approaches of close graph of mappings.

Further S. Hussain and B. Ahmad [1-6] continued studying the properties of γ-open (closed) sets and generalized many classical notions in their work. The purpose of this paper is to continue studying the properties of γ-regular open sets introduced and explored in [6]. The concept of γ-closed spaces have also been defined and discussed.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. PRELIMINARIES

Throughout the present paper, X denotes topological spaces.

Definition 2.1. [7] An operation $\gamma : \tau \to$ P(X) is a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.

Definition 2.2. [7] Let $A \subseteq X$. A point $x \in A$ is said to be γ-interior point of A, if there exists an open nbd N of x such that $N^\gamma \subseteq$ A and we denote the set of all such points by int_γ(A). Thus

$$int_\gamma\,(\mathrm{A}) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ-open [8] iff A = int_γ(A). A set A is called γ- closed [1] iff X-A is γ-open.

Definition 2.3. [1] A point $x \in X$ is called a γ-closure point of A$\subseteq$ X, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by cl_γ(A). A subset A of X is called γ-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every γ-closed superset of A.

Definition 2.4. [7] An operation γ on τ is said be regular, if for any open nbds U, V of x $\in$ X, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

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Definition 2.5. [8] An operation γ on τ is said to be open, if for any open nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^{\gamma} \supseteq B$.

3. γ-Regular-Open Sets

Definition 3.6. [6] A subset A of X is said to be γ-regular-open (resp. γ-regular-closed), if $A = int_{\gamma}(cl_{\gamma}(A))$ (resp. $A = cl_{\gamma}(int_{\gamma}(A))$).

It is clear that $RO_{\gamma}(X, \tau) \subseteq \tau_{\gamma} \subseteq \tau$ [6].

The following example shows that the converse of above inclusion is not true in general.

Example 3.1. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A \end{cases}$$

Calculations shows that $\{a, b\}, \{a, c\}, \{b\}, X, \phi$ are γ-open sets and $\{a, c\}, \{b\}, X, \phi$ are γ-regular-open sets. Here set $\{a, b\}$ is γ-open but not γ-regular-open.

Definition 3.7. [7] A space X is called γ-extremally disconnected, if for all γ-open subset U of X, $cl_{\gamma}(U)$ is a γ-open subset of X.

Proposition 3.1. *If A is a γ-clopen set in X, then A is a γ-regular-open set. Moreover, if X is γ-extremally disconnected then the converse holds.*

Proof. If A is a γ-clopen set, then $A = cl_{\gamma}(A)$ and $A = int_{\gamma}(A)$, and so we have $A = int_{\gamma}(cl_{\gamma}(A))$. Hence A is γ-regular-open.

Suppose that X is a γ-extremally disconnected space and A is a γ-regular-open set in X. Then A is γ-open and so $cl_{\gamma}(A)$ is a γ-open set. Hence $A = int_{\gamma}(cl_{\gamma}(A)) = cl_{\gamma}(A)$ and hence A is γ-closed set. This completes the proof. $\square$

The following example shows that space X to be γ-extremally disconnected is necessary in the converse of above Proposition 3.1.

Example 3.2. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Define an operation $\gamma : \tau \to P(X)$ by $\gamma(B) = int(cl(B))$. Clearly X is not γ-extremally disconnected space. Calculations shows that $\{a\}, \{a, b\}, \{b\}, X, \phi$ are γ-open as well as γ-regular-open sets. Here $\{a\}$ is a γ-regular-open set but not γ-clopen set.

Theorem 3.1. *Let $A \subseteq X$, then $(a) \Rightarrow (b) \Rightarrow (c)$, where :*
(a) A is γ-clopen.
(b) $A = cl_{\gamma}(int_{\gamma}(A))$.
(c) $X - A$ is γ-regular-open.

Proof. $(a) \Rightarrow (b)$. This is obvious.

$(b) \Rightarrow (c)$. Let $A = cl_{\gamma}(int\gamma(A))$. Then $X - A = X - cl_{\gamma}(int\gamma(A)) = int_{\gamma}(X - int_{\gamma}(A)) = int_{\gamma}(cl_{\gamma}(X - A))$, and hence $X - A$ is γ-regular-open set. Hence the proof. $\square$

Using Proposition 3.1, we have the following theorem:

Theorem 3.2. *If X is a γ-extremally disconnected space. Then $(a) \Rightarrow (b) \Rightarrow (c)$, where:*
(a) $X - A$ is γ-regular-open.
(b) A is γ-regular-open.
(c) A is γ-clopen.

Proof. $(a) \Rightarrow (b)$. Suppose X is γ-extremally disconnected space. From proposition 3.1, $X - A$ is a γ-open and γ-closed set, and hence A is a γ-open and γ-closed set. Thus $A = int_\gamma(cl_\gamma(A))$ implies A is γ-regular-open set.

$(b) \Rightarrow (c)$. This directory follows from Proposition 3.1. This completes as required. □

Combining Theorems 3.1 and 3.2, we have the following:

Theorem 3.3. *If X is a γ-extremally disconnected space. Then the following statements are equivalent:*
(a) A is γ-clopen.
(b) $A = cl_\gamma(int_\gamma(A))$.
(c) $X - A$ is γ-regular-open.
(d) A is γ-regular-open.

Theorem 3.4. *Let $A \subseteq X$ and γ be an open operation. If $cl_\gamma(A)$ is a γ-regular-open set. Then A is a γ-open set in X. Moreover, if X is extremally γ-disconnected then the converse holds.*

Proof. Suppose that $cl_\gamma(A)$ is a γ-regular-open sets. Since γ is open, we have $A \subseteq cl_\gamma(A) \subseteq int_\gamma(cl_\gamma(cl_\gamma(A))) = int_\gamma(cl_\gamma(A)) = int_\gamma(A)$. This implies that A is γ-open set.

Suppose that X is γ-extremally disconnected and A is γ-open set. Then $cl_\gamma(A)$ is a γ-open set, and hence γ-clopen set. Thus by Theorem 3.3, $cl_\gamma(A)$ is a γ-regular-open set. This completes the proof. □

Corollary 3.1. *Let X be a γ-extremally disconnected space. Then for each subset A of X, the set $cl_\gamma(int_\gamma(A))$ is γ-regular-open sets.*

Definition 3.8. A point $x \in X$ is said to be a γ-θ-cluster point of a subset A of X, if $cl_\gamma(U) \cap A \neq \phi$ for every γ-open set U containing x. The set of all γ-θ-cluster points of A is called the γ-θ-closure of A and is denoted by $\gamma cl_\theta(A)$.

Definition 3.9. A subset A of X is said to be γ-θ-closed, if $\gamma cl_\theta(A) = A$. The complement of γ-θ-closed set is called γ-θ-open sets. Clearly a γ-θ-closed (γ-θ-open) is γ-closed(γ-open) set.

Proposition 3.2. *Let A and B be subsets of a space X. Then the following properties hold:*
(1) If $A \subseteq B$, then $\gamma cl_\theta(A) \subseteq \gamma cl_\theta(B)$.
(2) If A_i is γ-θ-closed in X, for each $i \in I$, then $\bigcap_{i\in I} A_i$ is γ-θ-closed in X.

Proof. This is obvious.
(2). Let A_i be γ-θ-closed in X for each $i \in I$. Then $A_i = \gamma cl_\theta(A_i)$ for each $i \in I$. Thus we have $\gamma cl_\theta(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} \gamma cl_\theta(A_i) = \bigcap_{i\in I} A_i \subseteq \gamma cl_\theta(\bigcap_{i\in I} A_i)$.
Therefore, we have $\gamma cl_\theta(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} A_i$ and hence $\bigcap_{i\in I} A_i$ is γ-θ-closed. Hence the proof. □

Theorem 3.5. *If γ is an open operation. Then for any subset A of γ-extremally disconnected space X, the following hold:*

$$\begin{aligned} \gamma cl_\theta(A) &= \bigcap\{V : A \subseteq V \text{ and } V \text{ is } \gamma\text{-}\theta\text{-closed}\} \\ &= \bigcap\{V : A \subseteq V \text{ and } V \text{ is } \gamma\text{-regular-open}\} \end{aligned}$$

Proof. Let $x \notin \gamma cl_\theta(A)$. Then there is a γ-open set V with $x \in V$ such that $cl_\gamma(V) \cap A = \phi$. By Theorem 3.4, $X - cl_\gamma(V)$ is γ-regular-open and hence $X - cl_\gamma(V)$ is a γ-θ-closed set containing A and $x \notin X - \gamma cl_\theta(V)$. Thus we have $x \notin \bigcap\{V : A \subseteq V$ and V is γ-θ-closed$\}$.

Conversely, suppose that $x \notin \bigcap\{\text{V} : A \subseteq V$ and V is γ-θ-closed$\}$. Then there exists a γ-θ-closed set V such that $A \subseteq V$ and $x \notin V$, and so there exists a γ-open set U with $x \in U$ such that $U \subseteq cl_\gamma(U) \subseteq X - V$. Thus we have $cl_\gamma(U) \cap A \subseteq cl_\gamma(U) \cap V = \phi$ implies $x \notin \gamma cl_\theta(A)$.
The proof of the second equation follows similarly. This completes the proof. □

Theorem 3.6. *Let γ be an open operation. If X is a γ-extremally disconnected space and $A \subseteq X$. Then the followings hold:*
(a) *$x \in \gamma cl_\theta(A)$ if and only if $V \cap A \neq \phi$, for each γ-regular-open set V with $x \in V$.*
(b) *A is γ-θ-open if and only if for each $x \in A$ there exists a γ-regular-open set V with $x \in V$ such that $V \subseteq A$.*
(c) *A is a γ-regular-open set if and only if A is γ-θ-clopen.*

Proof. (a) and (b) follows directly from Theorems 3.3 and 3.4.
(c) Let A be a γ-regular-open set. Then A is a γ-open set and so $A = cl_\gamma(A) = \gamma cl_\theta(A)$ and hence A is γ-θ-closed. Since $X - A$ is a γ-regular-open set, by the argument above, $X - A$ is γ-θ-closed and A is γ-θ-open. The converse is obvious. Hence the proof. □

It is obvious that γ-regular-open $\Rightarrow$ γ-θ-open $\Rightarrow$ γ-open. But the converses are not necessarily true as the following examples show.

Example 3.3. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = \begin{cases} A, & \text{if } b \in A \\ cl(A), & \text{if } b \notin A \end{cases}$$

Calculations shows that $\{a, b\}, \{a, c\}, \{b\}, X, \phi$ are γ-open sets as well as γ-θ-open sets and γ-regular-open sets are $\{a, c\}, \{b\}, X, \phi$. Then the subset $\{a, b\}$ is γ-θ-open but not γ-regular-open.

Example 3.4. Let X= $\{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ be a topology on X. For $b \in X$, define an operation $\gamma : \tau \to P(X)$ by

$$\gamma(A) = A^\gamma = \begin{cases} cl(A), & \text{if } b \in A \\ A, & \text{if } b \notin A \end{cases}$$

Calculations shows that $\{\phi, X, \{a\}, \{a, c\}\}$ are γ-open sets and $\{\phi, X, \{a, c\}\}$ are γ-θ-open sets. The the subset $\{a\}$ is γ-open but not γ-θ-open.

4. γ-Closed Spaces

Definition 4.10. A filterbase Γ in X, γ-R-converges to $x_0 \in X$, if for each γ-regular-open set A with $x_0 \in A$, there exists $F \in \Gamma$ such that $F \subseteq A$.

Definition 4.11. A filterbase Γ in X γ-R-accumulates to $x_0 \in X$, if for each γ-regular-open set A with $x_0 \in A$ and each $F \in \Gamma$, $F \cap A \neq \phi$.

The following theorems are direct consequence of the above definitions.

Theorem 4.7. *If a filterbase Γ in X, γ-R-converges to $x_0 \in X$, then Γ γ-R-accumulates to x_0.*

Theorem 4.8. *If Γ_1 and Γ_2 are filterbases in X such that Γ_2 subordinate to Γ_1 and Γ_2 γ-R-accumulates to x_0, then Γ_1 γ-R-accumulates to x_0.*

Theorem 4.9. *If Γ is a maximal filterbase in X, then Γ γ-R-accumulates to x_0 if and only if Γ γ-R-converges to x_0.*

Definition 4.12. A space X is said to be γ-closed, if every cover $\{V_\alpha : \alpha \in I\}$ of X by γ-open sets has a finite subset I_0 of I such that $X = \bigcup_{\alpha\in I_0} cl_\gamma(V_\alpha)$.

Proposition 4.3. *If γ is an open operation, then the following are equivalent:*
(1) X is γ-closed.
(2) For each family $\{A_\alpha : \alpha \in I\}$ of γ-closed subsets of X such that $\bigcap_{\alpha\in I} A_\alpha = \phi$, there exists a finite subset I_0 of I such that $\bigcap_{\alpha\in I_0} int_\gamma(A_\alpha) = \phi$.
(3) For each family $\{A_\alpha : \alpha \in I\}$ of γ-closed subsets of X, if $\bigcap_{\alpha\in I_0} int_\gamma(A_\alpha) \neq \phi$, for every finite subset I_0 of I, then $\bigcap_{\alpha\in I} A_\alpha \neq \phi$.
(4) Every filterbase Γ in X γ-R-accumulates to $x_0 \in X$.
(5) Every maximal filterbase Γ in X γ-R-converges to $x_0 \in X$.

Proof. (2) $\Leftrightarrow$ (3). This is obvious.

(2) $\Rightarrow$ (1). Let $\{A_\alpha : \alpha \in I\}$ be a family of γ-open subsets of X such hat $X = \bigcup_{\alpha\in I} A_\alpha$. Then each $X - A_\alpha$ is a γ-closed subset of X and $\bigcap_{\alpha\in I}(X - A_\alpha) = \phi$, and so there exists a finite subset I_0 of I such that $\bigcap_{\alpha\in I_0} int_\gamma(X - A_\alpha) = \phi$, and hence $X = \bigcup_{\alpha\in I_0}(X - int_\gamma(X - A_\alpha)) = \bigcup_{\alpha\in I_0} cl_\gamma(A_\alpha)$. Therefore X is γ-closed, since γ is open.

(4) $\Rightarrow$ (2). Let $\{A_\alpha : \alpha \in I\}$ be a family of γ-closed subsets of X such that $\bigcap_{\alpha\in I} A_\alpha = \phi$. Suppose that for every finite subfamily $\{A_{\alpha_i} : i = 1, 2, ..., n\}$, $\bigcap_{i=1}^n int_\gamma(A_{\alpha_i}) \neq \phi$. Then $\bigcap_{i=1}^n(A_{\alpha_i}) \neq \phi$ and $\Gamma = \{\bigcap_{i=1}^n A_{\alpha_i} : n \in N, \alpha_i \in I\}$ forms a filterbase in X. By (4), Γ γ-R-accumulates to some $x_0 \in X$. Thus for every γ-open set A with $x_0 \in A$ and every $F \in \Gamma$, $F \cap cl_\gamma(A) \neq \phi$. Since $\bigcap_{F\in\Gamma} F = \phi$, there exists a $F \in \Gamma$ such that $x_0 \notin F$, and so there exists $\alpha_0 \in I$ such that $x_0 \notin A_{\alpha_0}$ and hence $x_0 \in X - A_{\alpha_0}$ and $X - A_{\alpha_0}$ is a γ-open set. Thus $x_0 \notin int_\gamma(A_{\alpha_0})$ and $x_0 \in X - int_\gamma(A_{\alpha_0})$, and hence $F_0 \cap (X - int_\gamma(A_{\alpha_0})) = F_0 \cap cl_\gamma(X - A_{\alpha_0}) = \phi$, which is a contradiction to our hypothesis.

(5) $\Rightarrow$ (4). Let Γ be filterbase in X. Then there exists a maximal filterbase ξ in X such that ξ subordinate to Γ. Since ξ γ-R-converges to x_0, so by Theorems 4.8 and 4.9, Γ γ-R-accumulate to x_0 .

(1) $\Rightarrow$ (5). Suppose that $\Gamma = \{F_a : a \in I\}$ is a maximal filterbase in X which does not γ-R-converge to any point in X. From Theorem 4.9, Γ does not γ-R-accumulates at any point in X. Thus for every $x \in X$, there exists a γ-open set A_x containing x and $F_{a_x} \in \Gamma$ such that $F_{a_x} \cap cl_\gamma(A_x) = \phi$. Since $\{A_x : x \in X\}$ is γ-open cover of X, there exists a finite subfamily $\{A_{x_i} : i = 1, 2, ..., n\}$ such that $X = \bigcup_{i=1}^n cl_\gamma(A_{x_i})$. Because Γ is a filterbase in X, there exists $F_0 \in \Gamma$ such that $F_0 \subseteq \bigcap_{i=1}^n F_{a_{x_i}}$, and hence $F_0 \cap cl_\gamma(A_{x_i})) = \phi$ for all $i = 1, 2, ..., n$. Hence we have that, $\phi = F_0 \bigcap(\bigcup_{i=1}^n cl_\gamma(A_{x_i})) = F_0 \cap X$, and hence $F_0 = \phi$. This is a contradiction. Hence the proof. □

Definition 4.13. A net $(x_i)_{i\in D}$ in a space X is said to be γ-R-converges to $x \in X$, if for each γ-open set U with $x \in U$, there exists i_0 such that $x_i \in cl_\gamma(U)$ for all $i \geqslant i_0$, where D is a directed set.

Definition 4.14. A net $(x_i)_{i\in D}$ in a space X is said to be γ-R-accumulates to $x \in X$, if for each γ-open set U with $x \in U$ and each i, $x_i \in cl_\gamma(U)$, where D is a directed set.

The proofs of following propositions are easy and thus are omitted:

Proposition 4.4. *Let $(x_i)_{i\in D}$ be a net in X. For the filterbase $F((x_i)_{i\in D}) = \{\{x_i : i \leq j\} : j \in D\}$ in X,*
(1) $F((x_i)_{i\in D})$ γ-R-converges to x if and only if $(x_i)_{i\in D}$ γ-R-converges to x.
(2) $F((x_i)_{i\in D})$ γ-R-accumulates to x if and only if $(x_i)_{i\in D}$ γ-R-accumulates to x.

Proposition 4.5. *Every filterbase F in X determines a net $(x_i)_{i\in D}$ in X such that*
(1) *F γ-R-converges to x if and only if $(x_i)_{i\in D}$ γ-R-converges to x.*
(2) *F γ-R-accumulates to x if and only if $(x_i)_{i\in D}$ γ-R-accumulates to x.*

From Propositions 4.4 and 4.5, filterbases and nets are equivalent in the sense of γ-R-converges and γ-R-accumulates. Thus we have the following theorem:

Theorem 4.10. *For a space X, the following are equivalent:*
(1) *X is γ-closed.*
(2) *Each net $(x_i)_{i\in D}$ in X has a γ-R-accumulation point.*
(3) *Each universal net in X γ-R-converges.*

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