On $\gamma$-regular-open sets and $\gamma$-closed spaces

SABIR HUSSAIN

ABSTRACT. The purpose of this paper is to continue studying the properties of $\gamma$-regular open sets introduced and explored by S. Hussain and B. Ahmad in 2007. The concept of $\gamma$-closed spaces have also been defined and discussed.

1. INTRODUCTION

The concept of operation $\gamma$ was initiated by S. Kasahara [7]. He also introduced $\gamma$-closed graph of a function. Using this operation, H. Ogata [8] introduced the concept of $\gamma$-open sets and investigated the related topological properties of the associated topology $\tau_\gamma$ and $\tau$. He further investigated general operator approaches of close graph of mappings.

Further S. Hussain and B. Ahmad [1-6] continued studying the properties of $\gamma$-open (closed) sets and generalized many classical notions in their work. The purpose of this paper is to continue studying the properties of $\gamma$-regular open sets introduced and explored in [6]. The concept of $\gamma$-closed spaces have also been defined and discussed.

First, we recall some definitions and results used in this paper. Hereafter, we shall write a space in place of a topological space.

2. PRELIMINARIES

Throughout the present paper, $X$ denotes topological spaces.

**Definition 2.1.** [7] An operation $\gamma : \tau \to P(X)$ is a function from $\tau$ to the power set of $X$ such that $V \subseteq V^\gamma$, for each $V \in \tau$, where $V^\gamma$ denotes the value of $\gamma$ at $V$. The operations defined by $\gamma(G) = G$, $\gamma(G) = cl(G)$ and $\gamma(G) = intcl(G)$ are examples of operation $\gamma$.

**Definition 2.2.** [7] Let $A \subseteq X$. A point $x \in A$ is said to be $\gamma$-interior point of $A$, if there exists an open nbd $N$ of $x$ such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$. Thus

$$int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$  

Note that $A$ is $\gamma$-open [8] iff $A = int_\gamma(A)$. A set $A$ is called $\gamma$-closed [1] iff $X-A$ is $\gamma$-open.

**Definition 2.3.** [1] A point $x \in X$ is called a $\gamma$-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd $U$ of $x$. The set of all $\gamma$-closure points of $A$ is called $\gamma$-closure of $A$ and is denoted by $cl_\gamma(A)$. A subset $A$ of $X$ is called $\gamma$-closed, if $cl_\gamma(A) \subseteq A$. Note that $cl_\gamma(A)$ is contained in every $\gamma$-closed superset of $A$.

**Definition 2.4.** [7] An operation $\gamma$ on $\tau$ is said to be regular, if for any open nbds $U, V$ of $x \in X$, there exists an open nbd $W$ of $x$ such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.


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**Definition 2.5.** [8] An operation $\gamma$ on $\tau$ is said to be open, if for any open nbd $U$ of each $x \in X$, there exists $\gamma$-open set $B$ such that $x \in B$ and $U^\gamma \supseteq B$.

### 3. $\gamma$-Regular-Open Sets

**Definition 3.6.** [6] A subset $A$ of $X$ is said to be $\gamma$-regular-open (resp. $\gamma$-regular-closed), if $A = \text{int}_{\gamma}(\text{cl}_{\gamma}(A))$ (resp. $A = \text{cl}_{\gamma}(	ext{int}_{\gamma}(A))$).

It is clear that $RO_{\gamma}(X, \tau) \subseteq \tau_{\gamma} \subseteq \tau$ [6].

The following example shows that the converse of above inclusion is not true in general.

**Example 3.1.** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. For $b \in X$, define an operation $\gamma : \tau \rightarrow P(X)$ by

$$
\gamma(A) = \begin{cases} 
A, & \text{if } b \in A \\
\text{cl}(A), & \text{if } b \notin A
\end{cases}
$$

Calculations shows that $\{a, b\}, \{a, c\}, \{b\}, X, \phi$ are $\gamma$-open sets and $\{a, c\}, \{b\}, X, \phi$ are $\gamma$-regular-open sets. Here set $\{a, b\}$ is $\gamma$-open but not $\gamma$-regular-open.

**Definition 3.7.** [7] A space $X$ is called $\gamma$-extremely disconnected, if for all $\gamma$-open subset $U$ of $X$, $\text{cl}_{\gamma}(U)$ is a $\gamma$-open subset of $X$.

**Proposition 3.1.** If $A$ is a $\gamma$-clopen set in $X$, then $A$ is a $\gamma$-regular-open set. Moreover, if $X$ is $\gamma$-extremely disconnected then the converse holds.

**Proof.** If $A$ is a $\gamma$-clopen set, then $A = \text{cl}_{\gamma}(A)$ and $A = \text{int}_{\gamma}(A)$, and so we have $A = \text{int}_{\gamma}(\text{cl}_{\gamma}(A))$. Hence $A$ is $\gamma$-regular-open.

Suppose that $X$ is a $\gamma$-extremely disconnected space and $A$ is a $\gamma$-regular-open set in $X$. Then $A$ is $\gamma$-open and so $\text{cl}_{\gamma}(A)$ is a $\gamma$-open set. Hence $A = \text{int}_{\gamma}(\text{cl}_{\gamma}(A)) = \text{cl}_{\gamma}(A)$ and hence $A$ is $\gamma$-closed set. This completes the proof. $\square$

The following example shows that space $X$ to be $\gamma$-extremely disconnected is necessary in the converse of above Proposition 3.1.

**Example 3.2.** Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Define an operation $\gamma : \tau \rightarrow P(X)$ by $\gamma(B) = \text{int}(\text{cl}(B))$. Clearly $X$ is not $\gamma$-extremely disconnected space. Calculations shows that $\{a\}, \{a, b\}, \{b\}, X, \phi$ are $\gamma$-open as well as $\gamma$-regular-open sets. Here $\{a\}$ is a $\gamma$-regular-open set but not $\gamma$-clopen set.

**Theorem 3.1.** Let $A \subseteq X$, then (a) $\Rightarrow$ (b) $\Rightarrow$ (c), where:

(a) $A$ is $\gamma$-clopen.

(b) $A = \text{cl}_{\gamma}(\text{int}_{\gamma}(A))$.

(c) $X - A$ is $\gamma$-regular-open.

**Proof.** (a) $\Rightarrow$ (b). This is obvious.

(b) $\Rightarrow$ (c). Let $A = \text{cl}_{\gamma}(\text{int}_{\gamma}(A))$. Then $X - A = X - \text{cl}_{\gamma}(\text{int}_{\gamma}(A)) = \text{int}_{\gamma}(X - \text{int}_{\gamma}(A)) = \text{int}_{\gamma}(\text{cl}_{\gamma}(X - A))$, and hence $X - A$ is $\gamma$-regular-open set. Hence the proof. $\square$

Using Proposition 3.1, we have the following theorem:

**Theorem 3.2.** If $X$ is a $\gamma$-extremely disconnected space. Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c), where:

(a) $X - A$ is $\gamma$-regular-open.

(b) $A$ is $\gamma$-regular-open.

(c) $A$ is $\gamma$-clopen.
Proof. (a) ⇒ (b). Suppose \( X \) is \( \gamma \)-extremally disconnected space. From proposition 3.1, \( X - A \) is a \( \gamma \)-open and \( \gamma \)-closed set, and hence \( A \) is a \( \gamma \)-open and \( \gamma \)-closed set. Thus \( A = \text{int}_\gamma(\text{cl}_\gamma(A)) \) implies \( A \) is \( \gamma \)-regular-open set.

(b) ⇒ (c). This directory follows from Proposition 3.1. This completes as required. \( \square \)

Combining Theorems 3.1 and 3.2, we have the following:

**Theorem 3.3.** If \( X \) is a \( \gamma \)-extremally disconnected space. Then the following statements are equivalent:
(a) \( A \) is \( \gamma \)-clopen.
(b) \( A = \text{cl}_\gamma(\text{int}_\gamma(A)) \).
(c) \( X - A \) is \( \gamma \)-regular-open.
(d) \( A \) is \( \gamma \)-regular-open.

**Theorem 3.4.** Let \( A \subseteq X \) and \( \gamma \) be an open operation. If \( \text{cl}_\gamma(A) \) is a \( \gamma \)-regular-open set. Then \( A \) is a \( \gamma \)-open set in \( X \). Moreover, if \( X \) is extremally \( \gamma \)-disconnected then the converse holds.

Proof. Suppose that \( \text{cl}_\gamma(A) \) is a \( \gamma \)-regular-open sets. Since \( \gamma \) is open, we have \( A \subseteq \text{cl}_\gamma(A) \subseteq \text{int}_\gamma(\text{cl}_\gamma(\text{cl}_\gamma(A))) = \text{int}_\gamma(\text{cl}_\gamma(A)) = \text{int}_\gamma(A) \). This implies that \( A \) is \( \gamma \)-open set.

Suppose that \( X \) is \( \gamma \)-extremally disconnected and \( A \) is \( \gamma \)-open set. Then \( \text{cl}_\gamma(A) \) is a \( \gamma \)-open set, and hence \( \gamma \)-clopren set. Thus by Theorem 3.3, \( \text{cl}_\gamma(A) \) is a \( \gamma \)-regular-open set. This completes the proof. \( \square \)

**Corollary 3.1.** Let \( X \) be a \( \gamma \)-extremally disconnected space. Then for each subset \( A \) of \( X \), the set \( \text{cl}_\gamma(\text{int}_\gamma(A)) \) is \( \gamma \)-clopren sets.

**Definition 3.8.** A point \( x \in X \) is said to be a \( \gamma \)-\( \theta \)-cluster point of a subset \( A \) of \( X \), if \( \text{cl}_\gamma(U) \cap A \neq \phi \) for every \( \gamma \)-open set \( U \) containing \( x \). The set of all \( \gamma \)-\( \theta \)-cluster points of \( A \) is called the \( \gamma \)-\( \theta \)-closure of \( A \) and is denoted by \( \gamma \text{cl}_\theta(A) \).

**Definition 3.9.** A subset \( A \) of \( X \) is said to be \( \gamma \)-\( \theta \)-closed, if \( \gamma \text{cl}_\theta(A) = A \). The complement of \( \gamma \)-\( \theta \)-closed set is called \( \gamma \)-\( \theta \)-open sets. Clearly a \( \gamma \)-\( \theta \)-closed (\( \gamma \)-\( \theta \)-open) is \( \gamma \)-closed(\( \gamma \)-open) set.

**Proposition 3.2.** Let \( A \) and \( B \) be subsets of a space \( X \). Then the following properties hold:
(1) If \( A \subseteq B \), then \( \gamma \text{cl}_\theta(A) \subseteq \gamma \text{cl}_\theta(B) \).
(2) If \( A_i \) is \( \gamma \)-\( \theta \)-closed in \( X \), for each \( i \in I \), then \( \bigcap_{i \in I} A_i \) is \( \gamma \)-\( \theta \)-closed in \( X \).

Proof. This is obvious.

(2). Let \( A_i \) be \( \gamma \)-\( \theta \)-closed in \( X \) for each \( i \in I \). Then \( A_i = \gamma \text{cl}_\theta(A_i) \) for each \( i \in I \). Thus we have \( \gamma \text{cl}_\theta(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} \gamma \text{cl}_\theta(A_i) \).
Therefore, we have \( \gamma \text{cl}_\theta(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} A_i \) and hence \( \bigcap_{i \in I} A_i \) is \( \gamma \)-\( \theta \)-closed. Hence the proof. \( \square \)

**Theorem 3.5.** If \( \gamma \) is an open operation. Then for any subset \( A \) of \( \gamma \)-extremally disconnected space \( X \), the following hold:
\[ \gamma \text{cl}_\theta(A) = \bigcap \{ V : A \subseteq V \text{ and } V \text{ is } \gamma \text{-\( \theta \)-closed} \} \]
\[ = \bigcap \{ V : A \subseteq V \text{ and } V \text{ is } \gamma \text{-\( \theta \)-regular-open} \} \]

Proof. Let \( x \notin \gamma \text{cl}_\theta(A) \). Then there is a \( \gamma \)-open set \( V \) with \( x \in V \) such that \( \text{cl}_\gamma(V) \cap A = \phi \).
By Theorem 3.4, \( X - \text{cl}_\gamma(V) \) is \( \gamma \)-regular-open and hence \( X - \text{cl}_\gamma(V) \) is a \( \gamma \)-\( \theta \)-closed set containing \( A \) and \( x \notin X - \gamma \text{cl}_\theta(V) \). Thus we have \( x \notin \bigcap \{ V : A \subseteq V \text{ and } V \text{ is } \gamma \)-\( \theta \)-closed \}. 

Conversely, suppose that \( x \notin \bigcap \{ V : A \subseteq V \text{ and } V \text{ is } \gamma\theta\text{-closed} \} \). Then there exists a \( \gamma\theta\text{-closed set } V \text{ such that } A \subseteq V \text{ and } x \notin V \), and so there exists a \( \gamma\text{-open set } U \text{ with } x \in U \text{ such that } U \subseteq \text{cl}_\gamma(U) \subseteq X - V \). Thus we have \( \text{cl}_\gamma(U) \cap A \subseteq \text{cl}_\gamma(U) \cap V = \phi \) implies \( x \notin \gamma\text{cl}_\phi(A) \).

The proof of the second equation follows similarly. This completes the proof. \( \square \)

**Theorem 3.6.** Let \( \gamma \) be an open operation. If \( X \) is a \( \gamma\)-extremally disconnected space and \( A \subseteq X \). Then the followings hold:

(a) \( x \in \gamma\text{cl}_\phi(A) \) if and only if \( V \cap A \neq \phi \), for each \( \gamma\)-regular-open set \( V \) with \( x \in V \).

(b) \( A \) is \( \gamma\theta\)-open if and only if for each \( x \in A \) there exists a \( \gamma\)-regular-open set \( V \) with \( x \in V \) such that \( V \subseteq A \).

(c) \( A \) is a \( \gamma\)-regular-open set if and only if \( A \) is \( \gamma\theta\)-copen.

**Proof.** (a) and (b) follows directly from Theorems 3.3 and 3.4.

(c) Let \( A \) be a \( \gamma\)-regular-open set. Then \( A \) is a \( \gamma\)-open set and so \( A = \text{cl}_\gamma(A) = \gamma\text{cl}_\phi(A) \) and hence \( A \) is \( \gamma\theta\)-closed. Since \( X - A \) is a \( \gamma\)-regular-open set, by the argument above, \( X - A \) is \( \gamma\theta\)-closed and \( A \) is \( \gamma\theta\)-open. The converse is obvious. Hence the proof. \( \square \)

It is obvious that \( \gamma\)-regular-open \( \Rightarrow \gamma\theta\)-open \( \Rightarrow \gamma\)-open. But the converses are not necessarily true as the following examples show.

**Example 3.3.** Let \( X = \{ a, b, c \} \), \( \tau = \{ \phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\} \} \). For \( b \in X \), define an operation \( \gamma : \tau \rightarrow P(X) \) by

\[
\gamma(A) = \begin{cases} 
A, & \text{if } b \in A \\
\text{cl}(A), & \text{if } b \notin A 
\end{cases}
\]

Calculations shows that \( \{a, b\}, \{a, c\}, \{b\}, X, \phi \) are \( \gamma\)-open sets as well as \( \gamma\theta\)-open sets and \( \gamma\)-regular-open sets are \( \{a, c\}, \{b\}, X, \phi \). Then the subset \( \{a, b\} \) is \( \gamma\theta\)-open but not \( \gamma\)-regular-open.

**Example 3.4.** Let \( X = \{ a, b, c \} \), \( \tau = \{ \phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\} \} \) be a topology on \( X \). For \( b \in X \), define an operation \( \gamma : \tau \rightarrow P(X) \) by

\[
\gamma(A) = A^\gamma = \begin{cases} 
\text{cl}(A), & \text{if } b \in A \\
A, & \text{if } b \notin A 
\end{cases}
\]

Calculations shows that \( \{ \phi, X, \{a\}, \{a, c\} \} \) are \( \gamma\)-open sets and \( \{ \phi, X, \{a, c\} \} \) are \( \gamma\theta\)-open sets. The the subset \( \{a\} \) is \( \gamma\)-open but not \( \gamma\theta\)-open.

4. \( \gamma\)-Closed Spaces

**Definition 4.10.** A filterbase \( \Gamma \) in \( X \), \( \gamma\)-R-converges to \( x_0 \in X \), if for each \( \gamma\)-regular-open set \( A \) with \( x_0 \in A \), there exists \( F \in \Gamma \) such that \( F \subseteq A \).

**Definition 4.11.** A filterbase \( \Gamma \) in \( X \), \( \gamma\)-R-accumulates to \( x_0 \in X \), if for each \( \gamma\)-regular-open set \( A \) with \( x_0 \in A \) and each \( F \in \Gamma \), \( F \cap A \neq \phi \).

The following theorems are direct consequence of the above definitions.

**Theorem 4.7.** If a filterbase \( \Gamma \) in \( X \), \( \gamma\)-R-converges to \( x_0 \in X \), then \( \Gamma \) \( \gamma\)-R-accumulates to \( x_0 \).

**Theorem 4.8.** If \( \Gamma_1 \) and \( \Gamma_2 \) are filterbases in \( X \) such that \( \Gamma_2 \) subordinate to \( \Gamma_1 \) and \( \Gamma_2 \) \( \gamma\)-R-accumulates to \( x_0 \), then \( \Gamma_1 \) \( \gamma\)-R-accumulates to \( x_0 \).

**Theorem 4.9.** If \( \Gamma \) is a maximal filterbase in \( X \), then \( \Gamma \) \( \gamma\)-R-accumulates to \( x_0 \) if and only if \( \Gamma \) \( \gamma\)-R-converges to \( x_0 \).
Definition 4.12. A space $X$ is said to be $\gamma$-closed, if every cover $\{V_\alpha : \alpha \in I\}$ of $X$ by $\gamma$-open sets has a finite subset $I_0$ of $I$ such that $X = \bigcup_{\alpha \in I_0} \text{cl}_\gamma(V_\alpha)$.

Proposition 4.3. If $\gamma$ is an open operation, then the following are equivalent:

1. $X$ is $\gamma$-closed.
2. For each family $\{A_\alpha : \alpha \in I\}$ of $\gamma$-closed subsets of $X$ such that $\bigcap_{\alpha \in I} A_\alpha = \phi$, there exists a finite subset $I_0$ of $I$ such that $\bigcap_{\alpha \in I_0} \text{int}_\gamma(A_\alpha) = \phi$.
3. For each family $\{A_\alpha : \alpha \in I\}$ of $\gamma$-closed subsets of $X$, if $\bigcap_{\alpha \in I_0} \text{int}_\gamma(A_\alpha) \neq \phi$, for every finite subset $I_0$ of $I$, then $\bigcap_{\alpha \in I_0} A_\alpha \neq \phi$.
4. Every filterbase $\Gamma$ in $X$ $\gamma$-R-accumulates to $x_0 \in X$.
5. Every maximal filterbase $\Gamma$ in $X$ $\gamma$-R-converges to $x_0 \in X$.

Proof. (2) $\iff$ (3). This is obvious.

(2) $\implies$ (1). Let $\{A_\alpha : \alpha \in I\}$ be a family of $\gamma$-open subsets of $X$ such that $X = \bigcup_{\alpha \in I} A_\alpha$. Then each $X - A_\alpha$ is a $\gamma$-closed subset of $X$ and $\bigcap_{\alpha \in I} (X - A_\alpha) = \phi$, and so there exists a finite subset $I_0$ of $I$ such that $\bigcap_{\alpha \in I_0} \text{int}_\gamma(X - A_\alpha) = \phi$, and hence $X = \bigcup_{\alpha \in I_0} (X - \text{int}_\gamma(X - A_\alpha)) = \bigcup_{\alpha \in I_0} \text{cl}_\gamma(A_\alpha)$. Therefore $X$ is $\gamma$-closed, since $\gamma$ is open.

(4) $\implies$ (2). Let $\{A_\alpha : \alpha \in I\}$ be a family of $\gamma$-closed subsets of $X$ such that $\bigcap_{\alpha \in I} A_\alpha = \phi$. Suppose that for every finite subfamily $\{A_{\alpha_i} : i = 1, 2, \ldots, n\}, \bigcap_{i=1}^n \text{int}_\gamma(A_{\alpha_i}) \neq \phi$. Then $\bigcap_{i=1}^n (A_{\alpha_i}) = \phi$ and $\Gamma = \big\{ \bigcap_{i=1}^n A_{\alpha_i} : n \in \mathbb{N}, \alpha_i \in I \big\}$ forms a filterbase in $X$. By (4), $\Gamma$ $\gamma$-R-accumulates to some $x_0 \in X$. Thus for every $\gamma$-open set $A$ with $x_0 \in A$ and every $F \in \Gamma$, $F \cap \text{cl}_\gamma(A) \neq \phi$. Since $\bigcap_{F \in \Gamma} F = \phi$, there exists a $F \in \Gamma$ such that $x_0 \notin F$, and so there exists $\alpha_0 \in I$ such that $x_0 \notin A_{\alpha_0}$ and hence $x_0 \in X - A_{\alpha_0}$ and $X - A_{\alpha_0}$ is a $\gamma$-open set. Thus $x_0 \notin \text{int}_\gamma(A_{\alpha_0})$ and $x_0 \in X - \text{int}_\gamma(A_{\alpha_0})$, and hence $F_0 \cap (X - \text{int}_\gamma(A_{\alpha_0})) = F_0 \cap \text{cl}_\gamma(X - A_{\alpha_0}) = \phi$, which is a contradiction to our hypothesis.

(5) $\implies$ (4). Let $\Gamma$ be filterbase in $X$. Then there exists a maximal filterbase $\xi$ in $X$ such that $\xi$ subordinate to $\Gamma$. Since $\xi$ $\gamma$-R-converges to $x_0$, so by Theorems 4.8 and 4.9, $\Gamma$ $\gamma$-R-accumulate to $x_0$.

(1) $\implies$ (5). Suppose that $\Gamma = \{F_\alpha : a \in I\}$ is a maximal filterbase in $X$ which does not $\gamma$-R-converge to any point in $X$. From Theorem 4.9, $\Gamma$ does not $\gamma$-R-accumulates at any point in $X$. Thus for every $x \in X$, there exists a $\gamma$-open set $A_x$ containing $x$ and $F_{ax} \in \Gamma$ such that $F_{ax} \cap \text{cl}_\gamma(A_x) = \phi$. Since $\{A_x : x \in X\}$ is $\gamma$-open cover of $X$, there exists a finite subfamily $\{A_{x_i} : i = 1, 2, \ldots, n\}$ such that $X = \bigcup_{i=1}^n \text{cl}_\gamma(A_{x_i})$. Because $\Gamma$ is a filterbase in $X$, there exists $F_0 \in \Gamma$ such that $F_0 \subseteq \bigcup_{i=1}^n F_{ax_i}$, and hence $F_0 \cap \text{cl}_\gamma(A_{x_i}) = \phi$ for all $i = 1, 2, \ldots, n$. Hence we have that, $\phi = F_0 \cap \bigcup_{i=1}^n \text{cl}_\gamma(A_{x_i}) = F_0 \cap X$, and hence $F_0 = \phi$. This is a contradiction. Hence the proof.

Definition 4.13. A net $(x_i)_{i \in D}$ in a space $X$ is said to be $\gamma$-R-converges to $x \in X$, if for each $\gamma$-open set $U$ with $x \in U$, there exists $i_0$ such that $x_i \in \text{cl}_\gamma(U)$ for all $i \geq i_0$, where $D$ is a directed set.

Definition 4.14. A net $(x_i)_{i \in D}$ in a space $X$ is said to be $\gamma$-R-accumulates to $x \in X$, if for each $\gamma$-open set $U$ with $x \in U$ and each $i, x_i \in \text{cl}_\gamma(U)$, where $D$ is a directed set.

The proofs of following propositions are easy and thus are omitted:

Proposition 4.4. Let $(x_i)_{i \in D}$ be a net in $X$. For the filterbase $F((x_i)_{i \in D}) = \{\{x_i : i \leq j\} : j \in D\}$ in $X$,

1. $F((x_i)_{i \in D})$ $\gamma$-R-converges to $x$ if and only if $(x_i)_{i \in D}$ $\gamma$-R-converges to $x$.
2. $F((x_i)_{i \in D})$ $\gamma$-R-accumulates to $x$ if and only if $(x_i)_{i \in D}$ $\gamma$-R-accumulates to $x$. 
Proposition 4.5. Every filterbase $F$ in $X$ determines a net $(x_i)_{i \in D}$ in $X$ such that
(1) $F \gamma$-$R$-converges to $x$ if and only if $(x_i)_{i \in D} \gamma$-$R$-converges to $x$.
(2) $F \gamma$-$R$-accumulates to $x$ if and only if $(x_i)_{i \in D} \gamma$-$R$-accumulates to $x$.

From Propositions 4.4 and 4.5, filterbases and nets are equivalent in the sense of $\gamma$-$R$-
converges and $\gamma$-$R$-accumulates. Thus we have the following theorem:

Theorem 4.10. For a space $X$, the following are equivalent:
(1) $X$ is $\gamma$-closed.
(2) Each net $(x_i)_{i \in D}$ in $X$ has a $\gamma$-$R$-accumulation point.
(3) Each universal net in $X$ $\gamma$-$R$-converges.

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DEPARTMENT OF MATHEMATICS
YANBU UNIVERSITY
P. O. BOX 31387, YANBU ALSINAIYAH
SAUDI ARABIA
E-mail address: sabiriub@yahoo.com