CREAT. MATH. INFORM.
20 (2011), No. 2, 147 - 156

Online version at http://creative-mathematics.ubm.ro/
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

The generalization of some results for Bernstein and Stancu operators

DAN MICLĂUŞ and PETRU I. BRAICA

ABSTRACT. In the present paper we generalize some results for Bernstein and Stancu operators. Firstly, we establish a relationship between two results concerning calculation of test functions by Bernstein operators. Secondly, using this relationship and some known results we prove in every case a Voronovskaja type theorem, the uniform convergence and the order of approximation for Bernstein and Stancu operators.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
The operators $B_n : C([0,1]) \to C([0,1])$ given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x)$ are the fundamental Bernstein's polynomials defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.2}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n\}$ and any $n \in \mathbb{N}$, are called Bernstein operators and were first introduced in [4].
In what follows, let the real parameters α, β be given, such that $0 \le \alpha \le \beta$. The operators $P_n^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$ defined by

$$P_n^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), \tag{1.3}$$

for any $x \in [0,1]$, any $k \in \{0,1,\ldots,n\}$ and any $n \in \mathbb{N}$, where $p_{n,k}(x)$ are the fundamental Bernstein's polynomials given at (1.2), are called Stancu operators [19].

Remark 1.1. More results and properties concerning (1.1) and (1.3) can be found also in monographs [2], [3].

The aim of this paper is to generalize some results for the presented operators. Firstly, we establish a general formula concerning calculation of the test functions by Bernstein operators and next, taking this into account we will prove a Voronovskaja type theorem in every case for Bernstein and Stancu operators. Using some known results, which will be cited at the adequate moment we shall prove the uniform convergence, general Voronovskaja type formulas and the order of approximation up to twice continuously differentiable function for Bernstein type operators.

Received: 26.02.2011. In revised form: 28.09.2011. Accepted: 30.09.2011.
2010 *Mathematics Subject Classification.* 41A10, 41A25, 41A36.
Key words and phrases. *Bernstein operators, Stancu operators, falling factorials, Pochhammer symbol, Stirling numbers of second kind, Voronovskaja type theorem, modulus of continuity, order of approximation.*

2. Preliminaries

Of the greatest utility in the calculus of finite differences, in number theory, in the summation of series, in the calculation of the Bernstein polynomials, some numbers are introduced in 1730 by J. Stirling in his *Methodus differentialis* [20], subsequently called "Stirling numbers" of the first and second kind.

For any $x \in \mathbb{R}$ and any $n \in \mathbb{N}_0$, let $(x)_n := \prod_{i=0}^{n-1}(x-i)$, where $(x)_0 := 1$ be the falling factorial denoted by Pochhammer symbol. It is well known that

$$x^j = \sum_{i=0}^{j} S(j,i)(x)_i \tag{2.4}$$

holds, for any $x \in \mathbb{R}$ and any $j \in \mathbb{N}_0$, where $S(j,i)$ are the Stirling numbers of second kind. Now, let $i, j \in \mathbb{N}_0$ be natural numbers, then the Stirling numbers of second kind have the following properties:

$$S(j,i) := \begin{cases} 1, & \text{if } j=i=0;\ j=i \ \text{ or } \ j>1, i=1 \\ 0, & \text{if } j>0, i=0 \\ 0, & \text{if } j<i \\ i \cdot S(j-1,i) + S(j-1,i-1), & \text{if } j,i>1. \end{cases} \tag{2.5}$$

Let $e_j(x) = x^j$, with $j \in \mathbb{N}_0$ and $x \in [0,1]$ be the test functions.

The main result established in [14], by O. T. Pop and M. Farcaş concerning calculation of the test functions in general case by Bernstein operators is given by the following:

Proposition 2.1. [14] *If* $n, j \in \mathbb{N}$, *then*

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=1}^{j} S(j,i)(n)_i x^i. \tag{2.6}$$

During the preparation of the present paper, making some researches we discovered that, the relation (2.6) had been proved earlier by S. Karlin and Z. Ziegler in [12]. As a special case, we can find the same relation in the article [1], where the asymptotic expansion of multivariate Bernstein polynomials on a simplex are considered. Later, in [17] the authors O. T. Pop, D. Bărbosu and P. I. Braica proved another result concerning calculation of the test functions by Bernstein operators. Before to mention the result we set $\binom{n}{k} = 0$ and $A_n^k = 0$, where A_n^k are arrangements of n taken k, for any $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}\backslash\{0,1,2,\ldots,n\}$.

Theorem 2.1. [17] *For any* $j, n \in \mathbb{N}$ *and any* $x \in [0,1]$, *the following holds*

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1} a_j^{(i)} A_n^{j-i} x^{j-i}, \tag{2.7}$$

where

$$a_j^{(i)} > 0, \ \ i = \overline{1, j-2}, \qquad a_j^{(0)} = a_j^{(j-1)} = 1 \tag{2.8}$$

and

$$a_{j+1}^{(i)} = (j-i+1)a_j^{(i-1)} + a_j^{(i)}, \quad \textit{for} \quad 1 \le i \le j-1. \tag{2.9}$$

Corollary 2.1. [17] *If $j \in \mathbb{N}$ and $i \in \{1, 2, \dots, j-1\}$, then the coefficients $a_j^{(i)}$, which appear in* (2.7) *are expressed by*

$$a_j^{(i)} = \sum_{k_i=1}^{j-i} k_i \cdot \sum_{k_{i-1}=1}^{k_i} k_{i-1} \dots \sum_{k_3=1}^{k_4} k_3 \cdot \sum_{k_2=1}^{k_3} k_2 \cdot \sum_{k_1=1}^{k_2} k_1. \tag{2.10}$$

In this section we recall some results from [15] and [16], which we shall use in the present paper. Let I, J be real intervals and $I \cap J \neq \emptyset$. For any $n, k \in \mathbb{N}_0$, $n \neq 0$ consider the functions $\varphi_{n,k} : J \to \mathbb{R}$, with the property that $\varphi_{n,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{n,k} : E(I) \to \mathbb{R}$.
For any $n \in \mathbb{N}$ define the operator $L_n : E(I) \to F(J)$, by

$$L_n(f; x) = \sum_{k=0}^{n} \varphi_{n,k}(x) A_{n,k}(f), \tag{2.11}$$

where $E(I)$ is a linear space of real-valued functions defined on I and $F(J)$ is a subset of the set of real-valued functions defined on J.

Remark 2.2. [15] The operators $(L_n)_{n\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define $T^*_{n,i}$ by

$$T^*_{n,i}(L_n; x) = n^i L_n(\psi_x^i; x) = n^i \sum_{k=0}^{n} \varphi_{n,k}(x) A_{n,k}(\psi_x^i), \quad x \in I \cap J, \tag{2.12}$$

where $\psi_x^i(t) = (t-x)^i,\ t \in I \cap J$.
In what follows $s \in \mathbb{N}_0$ is even and we assume that the next two conditions hold:

- there exists the smallest $\alpha_s,\ \alpha_{s+2} \in [0, +\infty)$, so that

$$\lim_{n\to\infty} \frac{T^*_{n,j}(L_n; x)}{n^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{2.13}$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \tag{2.14}$$

- $I \cap J$ is an interval.

Theorem 2.2. [15, 16] *If $f \in E(I)$ is a function s times differentiable in a neighborhood of $x \in I \cap J$, then*

$$\lim_{n\to\infty} n^{s-\alpha_s} \left(L_n(f; x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i! \cdot n^i} T^*_{n,i}(L_n; x) \right) = 0. \tag{2.15}$$

Assume that f is s times differentiable function on I and there exists an interval $K \subseteq I \cap J$, such that, there exist $n(s) \in \mathbb{N}$ and the constants $k_j \in \mathbb{R}$ depending on K, so that for $n \geq n(s)$ and $x \in K$, the following inequality

$$\frac{T^*_{n,j}(L_n; x)}{n^{\alpha_j}} \leq k_j, \tag{2.16}$$

holds, for $j \in \{s, s+2\}$.
Then, the convergence expressed by (2.15) *is uniform on* K *and moreover*

$$n^{s-\alpha_s}\left|L_n(f;x) - \sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!\cdot n^i}T^*_{n,i}(L_n;x)\right| \tag{2.17}$$
$$\leq \frac{1}{s!}(k_s + k_{s+2})\omega_1\left(f^{(s)};\frac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

for any $x \in K$ *and* $n \geq n(s)$*, where* $\omega_1(f;\delta)$ *denotes the modulus of continuity of the function* f.

3. Main results

Firstly, we want to get an answer to the following question:
Is it possible to exist a relationship between (2.6) and (2.7)? By comparing both the relations we can get an answer.

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=1}^{j}S(j,i)(n)_i x^i$$
$$= \frac{1}{n^j}\left(S(j,1)(n)_1 x + S(j,2)(n)_2 x^2 + \ldots + S(j,j-1)(n)_{j-1}x^{j-1} + S(j,j)(n)_j x^j\right)$$
$$= \frac{1}{n^j}\sum_{i=1}^{j}S(j,j+1-i)(n)_{j+1-i}x^{j+1-i} = \frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)(n)_{j-i}x^{j-i}.$$

In conclusion

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)(n)_{j-i}x^{j-i} \tag{3.18}$$

holds. One observes that

$$A_n^{j-i} = \frac{n!}{(n-(j-i))!} = n(n-1)\cdot\ldots\cdot(n-(j-i-1)) = (n)_{j-i}. \tag{3.19}$$

Taking (3.19) into account, it follows that the coefficients $a_j^{(i)}$ and $S(j,j-i)$ of x^{j-i} from (2.7) and (3.18) are equal, for any $j \in \mathbb{N}$ and $i \in \{0,1,\ldots,j-1\}$.

Proposition 3.2. *For any* $j, n \in \mathbb{N}$ *and any* $x \in [0,1]$*, the following holds:*

$$B_n(e_j;x) = \frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)(n)_{j-i}x^{j-i}. \tag{3.20}$$

The Stirling numbers which appear in (3.20) admit the following representation:

Corollary 3.2. *If* $j \in \mathbb{N}$ *and* $i \in \{1,2,\ldots,j-1\}$*, then the coefficients* $S(j,j-i)$ *can be expressed by*

$$S(j,j-i) = \sum_{k_i=1}^{j-i}k_i\cdot\sum_{k_{i-1}=1}^{k_i}k_{i-1}\ldots\sum_{k_3=1}^{k_4}k_3\cdot\sum_{k_2=1}^{k_3}k_2\cdot\sum_{k_1=1}^{k_2}k_1.$$

Proof. It follows immediately, because of (2.10) $a_j^{(i)} = S(j,j-i)$, for any $j \in \mathbb{N}$ and $i \in \{0,1,2,\ldots,j-1\}$. □

Remark 3.3. In the following, we assume that the first three cases concerning calculation of the test functions by Bernstein, respectively Stancu operators are well known.

3.1. **Bernstein operators.** Using the construction form preliminaries, we assume that $I = J = [0,1]$, $E(I) = C([0,1])$, $F(J) = C([0,1])$. Then the functions $\varphi_{n,k} : [0,1] \to \mathbb{R}$ let be defined by $\varphi_{n,k}(x) := p_{n,k}(x)$, for any $x \in [0,1]$, any $n, k \in \mathbb{N}_0$, $n \neq 0$ and the functionals $A_{n,k} : C([0,1]) \to \mathbb{R}$ let be defined by $A_{n,k}(f) := f\left(\frac{k}{n}\right)$, for any $n, k \in \mathbb{N}_0$, $n \neq 0$.
In this case one obtains the Bernstein operators, with

$$T^*_{n,i}(B_n; x) = n^i \sum_{k=0}^{n} p_{n,k}(x) A_{n,k}(\psi_x^i) = n^i \sum_{k=0}^{n} p_{n,k}(x) \left(\frac{k}{n} - x\right)^i \tag{3.21}$$
$$= \sum_{k=0}^{n} p_{n,k}(x)(k - nx)^i =: T_{n,i}(x).$$

Application 3.1. For $j \in \{3,4\}$ we present the calculation of test functions by Bernstein operators, taking (3.20) into account.

Case 1. $j = 3$

$$\begin{aligned} B_n(e_3; x) &= \frac{1}{n^3} \sum_{i=0}^{2} S(3, 3-i)(n)_{3-i} x^{3-i} \\ &= \frac{1}{n^3} \left(S(3,3)(n)_3 x^3 + S(3,2)(n)_2 x^2 + S(3,1)(n)_1 x\right) \\ &= \frac{1}{n^3} \left(n(n-1)(n-2)x^3 + 3n(n-1)x^2 + nx\right), \end{aligned}$$

where $S(3,2) = 2 \cdot S(2,2) + S(2,1) = 3$.

Case 2. $j = 4$

$$B_n(e_4; x) = \frac{1}{n^4} \sum_{i=0}^{3} S(4, 4-i)(n)_{4-i} x^{4-i}$$
$$= \frac{1}{n^4} \left(S(4,4)(n)_4 x^4 + S(4,3)(n)_3 x^3 + S(4,2)(n)_2 x^2 + S(4,1)(n)_1 x\right)$$
$$= \frac{1}{n^4} \left(n(n-1)(n-2)(n-3)x^4 + 6n(n-1)(n-2)x^3 + 7n(n-1)x^2 + nx\right),$$

where $S(4,2) = 2 \cdot S(3,2) + S(3,1) = 7$ and $S(4,3) = 3 \cdot S(3,3) + S(3,2) = 6$.

Remark 3.4. Concerning the polynomials $T_{n,i}(x) = T^*_{n,i}(B_n; x)$, which were first introduced in [13], we shall give a proof relied on Application 3.1.

Lemma 3.1. *For any $x \in [0,1]$ and any $n \in \mathbb{N}$, the following hold:*
$T^*_{n,0}(B_n; x) = 1,$
$T^*_{n,1}(B_n; x) = 0,$
$T^*_{n,2}(B_n; x) = nx(1-x),$
$T^*_{n,3}(B_n; x) = nx(1-x)(1-2x),$
$T^*_{n,4}(B_n; x) = 3n^2x^2(1-x)^2 + n\left(x(1-x) - 6x^2(1-x)^2\right).$

Proof. Taking into account (2.12), (3.21) and Application 3.1, we get:

$$T^*_{n,0}(B_n; x) = B_n(e_0; x) = 1;$$

$$T^*_{n,1}(B_n; x) = nB_n(\psi_x; x) = n(B_n(e_1; x) - xB_n(e_0; x)) = 0;$$

$$\begin{aligned} T^*_{n,2}(B_n; x) &= n^2 B_n\left(\psi_x^2; x\right) = n^2\left(B_n(e_2; x) - 2xB_n(e_1; x) + x^2 B_n(e_0; x)\right) \\ &= nx(1-x); \end{aligned}$$

$$T^*_{n,3}(B_n;x) = n^3 B_n\left(\psi_x^3;x\right)$$
$$= n^3\left(B_n(e_3;x) - 3xB_n(e_2;x) + 3x^2B_n(e_1;x) - x^3B_n(e_0;x)\right)$$
$$= nx(1-x)(1-2x);$$

$$T^*_{n,4}(B_n;x) = n^4 B_n\left(\psi_x^4;x\right)$$
$$= n^4\left(B_n(e_4;x) - 4xB_n(e_3;x) + 6x^2B_n(e_2;x) - 4x^3B_n(e_1;x) + x^4B_n(e_0;x)\right)$$
$$= 3n^2x^2(1-x)^2 + n\left(x(1-x) - 6x^2(1-x)^2\right).$$

□

Lemma 3.2. *For any $x \in [0,1]$, the following*

$$\lim_{n\to\infty} T^*_{n,0}(B_n;x) = 1, \tag{3.22}$$

$$\lim_{n\to\infty} \frac{T^*_{n,2}(B_n;x)}{n} = x(1-x), \tag{3.23}$$

$$\lim_{n\to\infty} \frac{T^*_{n,4}(B_n;x)}{n^2} = 3(x(1-x))^2 \tag{3.24}$$

hold, and there exist

$$T^*_{n,0}(B_n;x) = 1 = k_0, \tag{3.25}$$

$$\frac{T^*_{n,2}(B_n;x)}{n} \le \frac{1}{4} = k_2, \tag{3.26}$$

$$\frac{T^*_{n,4}(B_n;x)}{n^2} \le \frac{3}{16} = k_4, \tag{3.27}$$

for any $x \in [0,1]$ and any $n \in \mathbb{N}$.

Proof. The identities (3.22)–(3.24) follow immediately from Lemma 3.1, while (3.25)–(3.27) yield from (3.22)–(3.24). □

Theorem 3.3. *Let $f \in C([0,1])$ be a function. If $x \in [0,1]$ and f is s times differentiable in a neighborhood of x, then*

$$\lim_{n\to\infty} B_n(f;x) = f(x), \tag{3.28}$$

for $s = 0$;

$$\lim_{n\to\infty} n(B_n(f;x) - f(x)) = \frac{x(1-x)}{2} f^{(2)}(x), \tag{3.29}$$

for $s = 2$;

$$\lim_{n\to\infty} n^2\left(B_n(f;x) - f(x) - \frac{x(1-x)}{2n} f^{(2)}(x)\right) \tag{3.30}$$
$$= \frac{x(1-x)(1-2x)}{6} f^{(3)}(x) + \frac{(x(1-x))^2}{8} f^{(4)}(x)$$

for $s = 4$ and

$$\lim_{n\to\infty} n^{s-\alpha_s}\left(B_n(f;x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i!\cdot n^i} T^*_{n,i}(B_n;x)\right) = 0, \tag{3.31}$$

for $s \geq 4$.
Assume that f *is* s *times differentiable on* $[0,1]$. *Then the convergence from* (3.28)–(3.31) *is uniform on* $[0,1]$. *Moreover, we get*

$$|B_n(f;x) - f(x)| \leq \frac{5}{4} \cdot \omega_1 \left(f; \frac{1}{\sqrt{n}} \right), \tag{3.32}$$

for $s = 0$ *and*

$$n \left| B_n(f;x) - f(x) - \frac{x(1-x)}{2n} f^{(2)}(x) \right| \leq \frac{7}{32} \cdot \omega_1 \left(f^{(2)}; \frac{1}{\sqrt{n}} \right), \tag{3.33}$$

for $s = 2$.

Proof. It follows from Theorem 2.2, with $\alpha_0 = 0$, $\alpha_2 = 1$ and $\alpha_4 = 2$, taking into account Lemma 3.1 and Lemma 3.2. □

Remark 3.5. The above theorem, by the relation (3.31) generalizes the asymptotic behavior of the Bernstein operators and by some particular cases we recover formula (3.29), for twice continuously differentiable function, proved first by E. V. Voronovskaja [21], respectively formula (3.30), for four continuously differentiable function, proved first by S. N. Bernstein [5].
Concerning quantitative form of Voronovskaja result in terms of modulus of continuity we recover the well known estimate (3.32), obtained by G. G. Lorentz [13] and for twice continuously differentiable function we get a good estimate for a neighborhood of the point $\frac{1}{2}$. It is worth mentioning that the first (3.32) type estimation was obtained by T. Popoviciu [18]. Various quantitative forms of Voronovskaja's 1932 result dealing with the asymptotic behavior of the Bernstein operators are discussed in several recently papers [6], [7], [8], [9], [10] and [11], where better estimate close to the endpoints 0 and 1 then the global one was established.

3.2. **Stancu operators.** Using the same construction form preliminaries, if we assume that $I = J = [0,1]$, $E(I) = C([0,1])$, $F(J) = C([0,1])$, then the functions $\varphi_{n,k} : [0,1] \to \mathbb{R}$ are defined by $\varphi_{n,k}(x) := p_{n,k}(x)$, for any $x \in [0,1]$, any $n, k \in \mathbb{N}_0$, $n \neq 0$ and the functionals $A_{n,k} : C([0,1]) \to \mathbb{R}$ are defined by $A_{n,k}(f) := f\left(\frac{k+\alpha}{n+\beta}\right)$, for any $n, k \in \mathbb{N}_0$, $n \neq 0$.
In this case one obtains the Stancu operators, with

$$\begin{aligned}
T^*_{n,i}\left(P_n^{(\alpha,\beta)}; x\right) &= n^i \sum_{k=0}^{n} p_{n,k}(x) A_{n,k}(\psi_x^i) = n^i \sum_{k=0}^{n} p_{n,k}(x) \left(\frac{k+\alpha}{n+\beta} - x\right)^i \\
&= \left(\frac{n}{n+\beta}\right)^i \sum_{k=0}^{n} p_{n,k}(x)(k - nx + \alpha - \beta x)^i \\
&= \left(\frac{n}{n+\beta}\right)^i \sum_{k=0}^{n} p_{n,k}(x) \sum_{l=0}^{i} \binom{i}{l} (k-nx)^l (\alpha - \beta x)^{i-l} \\
&= \left(\frac{n}{n+\beta}\right)^i \sum_{l=0}^{i} \binom{i}{l} (\alpha - \beta x)^{i-l} T_{n,l}(x).
\end{aligned} \tag{3.34}$$

Lemma 3.3. *For any* $x \in [0,1]$ *and any* $n \in \mathbb{N}$, *the following hold:*
$T^*_{n,0}\left(P_n^{(\alpha,\beta)}; x\right) = 1$,

$T^*_{n,1}\left(P_n^{(\alpha,\beta)};x\right) = \frac{n}{n+\beta}(\alpha-\beta x)$,
$T^*_{n,2}\left(P_n^{(\alpha,\beta)};x\right) = \left(\frac{n}{n+\beta}\right)^2\left((\alpha-\beta x)^2+nx(1-x)\right)$,
$T^*_{n,3}\left(P_n^{(\alpha,\beta)};x\right) = \left(\frac{n}{n+\beta}\right)^3\left((\alpha-\beta x)^3+3(\alpha-\beta x)nx(1-x)+nx(1-x)(1-2x)\right)$,
$T^*_{n,4}\left(P_n^{(\alpha,\beta)};x\right) = \left(\frac{n}{n+\beta}\right)^4\left((\alpha-\beta x)^4+6(\alpha-\beta x)^2nx(1-x)\right.$
$\left.+4(\alpha-\beta x)nx(1-x)(1-2x)+3n^2x^2(1-x)^2+n\left(x(1-x)-6x^2(1-x)^2\right)\right)$.

Proof. Taking (3.34), (3.21) and Lemma 3.1 into account, we get:

$$T^*_{n,0}\left(P_n^{(\alpha,\beta)};x\right) = T_{n,0}(x) = 1;$$

$$T^*_{n,1}\left(P_n^{(\alpha,\beta)};x\right) = \frac{n}{n+\beta}\sum_{l=0}^{1}\binom{1}{l}(\alpha-\beta x)^{1-l}T_{n,l}(x) = \frac{n}{n+\beta}(\alpha-\beta x);$$

$$T^*_{n,2}\left(P_n^{(\alpha,\beta)};x\right) = \left(\frac{n}{n+\beta}\right)^2\sum_{l=0}^{2}\binom{2}{l}(\alpha-\beta x)^{2-l}T_{n,l}(x) =$$

$$\left(\frac{n}{n+\beta}\right)^2\left((\alpha-\beta x)^2+nx(1-x)\right);$$

$$T^*_{n,3}\left(P_n^{(\alpha,\beta)};x\right) = \left(\frac{n}{n+\beta}\right)^3\sum_{l=0}^{3}\binom{3}{l}(\alpha-\beta x)^{3-l}T_{n,l}(x)$$

$$= \left(\frac{n}{n+\beta}\right)^3\left((\alpha-\beta x)^3+3(\alpha-\beta x)nx(1-x)+nx(1-x)(1-2x)\right);$$

$$T^*_{n,4}\left(P_n^{(\alpha,\beta)};x\right) = \left(\frac{n}{n+\beta}\right)^4\sum_{l=0}^{4}\binom{4}{l}(\alpha-\beta x)^{4-l}T_{n,l}(x)$$

$$= \left(\frac{n}{n+\beta}\right)^4\left((\alpha-\beta x)^4+6(\alpha-\beta x)^2nx(1-x)\right.$$

$$\left.+4(\alpha-\beta x)nx(1-x)(1-2x)+3n^2x^2(1-x)^2+n\left(x(1-x)-6x^2(1-x)^2\right)\right).$$

□

Lemma 3.4. *For any $x\in[0,1]$, the following*

$$\lim_{n\to\infty} T^*_{n,0}\left(P_n^{(\alpha,\beta)};x\right) = 1, \tag{3.35}$$

$$\lim_{n\to\infty}\frac{T^*_{n,2}\left(P_n^{(\alpha,\beta)};x\right)}{n} = x(1-x), \tag{3.36}$$

$$\lim_{n\to\infty}\frac{T^*_{n,4}\left(P_n^{(\alpha,\beta)};x\right)}{n^2} = 3(x(1-x))^2 \tag{3.37}$$

hold, and there exist

$$T^*_{n,0}\left(P_n^{(\alpha,\beta)};x\right) = 1 = k_0, \tag{3.38}$$

$$\frac{T^*_{n,2}\left(P_n^{(\alpha,\beta)};x\right)}{n} \le \frac{1}{4} = k_2, \tag{3.39}$$

$$\frac{T^*_{n,4}\left(P_n^{(\alpha,\beta)};x\right)}{n^2} \le \frac{3}{16} = k_4, \tag{3.40}$$

for any $x \in [0,1]$ *and any* $n \in \mathbb{N}$.

Proof. The identities (3.35)–(3.37) follow immediately from Lemma 3.3, while (3.38)–(3.40) yield from (3.35)–(3.37). □

Theorem 3.4. *Let* $f \in C([0,1])$ *be a function. If* $x \in [0,1]$ *and* f *is* s *times differentiable in a neighborhood of* x*, then*

$$\lim_{n\to\infty} P_n^{(\alpha,\beta)}(f;x) = f(x), \tag{3.41}$$

for $s = 0$;

$$\lim_{n\to\infty} n\left(P_n^{(\alpha,\beta)}(f;x) - f(x)\right) = (\alpha - \beta x)f^{(1)}(x) + \frac{x(1-x)}{2}f^{(2)}(x), \tag{3.42}$$

for $s = 2$;

$$\lim_{n\to\infty} n^2\left(P_n^{(\alpha,\beta)}(f;x) - f(x) - \frac{\alpha-\beta x}{n+\beta}f^{(1)}(x) - \frac{(\alpha-\beta x)^2 + nx(1-x)}{2(n+\beta)^2}f^{(2)}(x)\right)$$
$$= \frac{3(\alpha-\beta x)x(1-x) + x(1-x)(1-2x)}{6}f^{(3)}(x) + \frac{(x(1-x))^2}{8}f^{(4)}(x), \tag{3.43}$$

for $s = 4$ *and*

$$\lim_{n\to\infty} n^{s-\alpha_s}\left(P_n^{(\alpha,\beta)}(f;x) - \sum_{i=0}^{s}\frac{f^{(i)}(x)}{i!\cdot n^i}T^*_{n,i}\left(P_n^{(\alpha,\beta)};x\right)\right) = 0, \tag{3.44}$$

for $s \ge 4$.
Assume that f *is* s *times differentiable on* $[0,1]$. *Then the convergence from* (3.41)–(3.44) *is uniform on* $[0,1]$. *Moreover, we get*

$$\left|P_n^{(\alpha,\beta)}(f;x) - f(x)\right| \le \frac{5}{4}\cdot\omega_1\left(f;\frac{1}{\sqrt{n}}\right), \tag{3.45}$$

for $s = 0$ *and*

$$n\left|P_n^{(\alpha,\beta)}(f;x) - f(x) - \frac{\alpha-\beta x}{n+\beta}f^{(1)}(x) - \frac{(\alpha-\beta x)^2 + nx(1-x)}{2(n+\beta)^2}f^{(2)}(x)\right|$$
$$\le \frac{7}{32}\cdot\omega_1\left(f^{(2)};\frac{1}{\sqrt{n}}\right), \tag{3.46}$$

for $s = 2$.

Proof. It follows from Theorem 2.2, with $\alpha_0 = 0$, $\alpha_2 = 1$ and $\alpha_4 = 2$, taking into account Lemma 3.3 and Lemma 3.4. □

Remark 3.6. The above theorem, by the relation (3.44) generalizes the asymptotic behavior of the Stancu operators and in the particular case, for $s = 2$ we recover formula (3.42), for twice continuously differentiable function, proved first by D. D. Stancu [19]. We also get the asymptotic behavior in the particular case $s = 4$ and quantitative forms in terms of modulus of continuity, for the same operators.

REFERENCES

[1] Abel, U. and Ivan, M., *Asymptotic expansion of the multivariate Bernstein polynomials on a simplex*, Approx. Theory and Appl. **16** (2000), No. 3, 85–93
[2] Agratini, O., *Approximation by linear operators (in Romanian)*, Presa Universitară Clujeană, Cluj-Napoca, 2000
[3] Altomare, F. and Campiti, M., *Korovkin-type Approximation Theory and its Applications*, de Gruyter Series Studies in Mathematics, Walter de Gruyter & Co. Berlin **17**, New York, 1994
[4] Bernstein, S. N., *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow **13** (1912-1913), No. 2, 1–2
[5] Bernstein, S. N., *Complément à l'article de E.V. Voronovskaja*, C. R. Acad. Sci. URSS (1932), 86–92
[6] Deo, N., Noor, M. A. and Siddiqui, M. A., *On approximation by a class of new Bernstein type operators*, Appl. Math. Comput. **201** (2008), No. 1-2, 604–612
[7] Deo, N. and Bhardwaj, N., *Some approximation theorems for multivariate Bernstein operators*, SEA. Bull. Math. **34** (2010), 1023–1034
[8] Gonska, H. H., Piţul, P. and Raşa, I., On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators, in *Proceedings of the International Conference on Numerical Analysis and Approximation Theory*, Cluj-Napoca, Romanian, July 5-8, 2006 (Agratini, O. and Blaga, P., Eds.), Casa Cărţii de Ştiinţă Publisher, Cluj-Napoca, 2006, 55–80
[9] Gonska, H. H., *On the degree of approximation in Voronovskaja's theorem*, Studia Univ. Babeş-Bolyai, Math. **52** (2007), No. 3, 103–115
[10] Gonska, H. H. and Raşa, I., *Remarks on Voronovskaja's theorem*, General Mathematics **16** (2008), No. 4, 87–97
[11] Gonska, H. H. and Tachev, G., *A quantitative variant of Voronovskaja's theorem*, Results in Math. **53** (2009), 287–294
[12] Karlin, S. and Ziegler, Z., *Iteration of positive approximation operators*, J. Approx. Theory **3** (1970), 310–339
[13] Lorentz, G. G., *Bernstein polynomials*, University of Toronto Press, Toronto, 1953
[14] Pop, O. T. and Farcaş, M., *About Bernstein polynomial and the Stirling's numbers of second type*, Creat. Math. Inform. **14** (2005), 53–56
[15] Pop, O. T., *The generalization of Voronovskaja's theorem for a class of linear and positive operators*, Rev. Anal. Num. Théor. Approx. **34** (2005), No. 1, 79–91
[16] Pop, O. T., *About a general property for a class of linear and positive operators and applications*, Rev. Anal. Num. Théor. Approx. **34** (2005), No. 2, 175–180
[17] Pop, O. T., Bărbosu, D. and Braica, P. I., *Some results regarding the Bernstein polynomials*, Acta Univ. Apulensis Math. Inform. **22** (2010), 243–247
[18] Popoviciu, T., *On the best approximation of continuous functions by polynomials (in Romanian)*, Inst. Arte Grafice Ardealul, Cluj-Napoca, 1937
[19] Stancu, D. D., *On a generalization of Bernstein polynomials (in Romanian)*, Studia Univ. Babeş-Bolyai, Ser. Math-Phys. **14** (1969), 31–45
[20] Stirling, J., *Differential Method: A Treatise of the Summation and Interpolation of Infinite Series (in Latin)*, London, 1730
[21] Voronovskaja, E. V., *Détérmination de la forme asymtotique d'approximation des fonctions par des polinômes de Bernstein*, C. R. Acad. Sci. URSS (1932), 79–85

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
NORTH UNIVERSITY OF BAIA MARE
VICTORIEI 76, 430122 BAIA MARE, ROMANIA
E-mail address: danmiclausrz@yahoo.com

SECONDARY SCHOOL "GRIGORE MOISIL"
MILENIULUI 1, 440037 SATU MARE, ROMANIA
E-mail address: petrubr@yahoo.com