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A refinement of the Kittaneh-Manasrah inequality

NICUŞOR MINCULETE

ABSTRACT. The purpose of this paper is to give refinement for the Kittaneh-Manasrah inequality which improves the inequality of Young. We also give several applications.

1. INTRODUCTION

Many results of Modern Mathematics are based on the theory of the inequalities. Improving the inequality of Kittaneh-Manasrah, we obtain an improvement of Young's inequality which improves other important inequalities such as the following: Bernoulli's inequality and the weighted power means.

Now, we present the famous Young inequality

$$\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda}, \tag{1.1}$$

for positive real numbers a, b and $\lambda \in [0,1]$. In [4], S. Furuichi given a reverse inequality for Young's inequality.

Inequality (1.1) was refined by F. Kittaneh and Y. Manasrah, in [7], thus

$$\lambda a + (1-\lambda)b \geq a^{\lambda} b^{1-\lambda} + r(\sqrt{a}-\sqrt{b})^2, \tag{1.2}$$

where $r = \min\{\lambda, 1-\lambda\}$. They use this inequality for the study of matrix norm inequalities.

Its reverse inequality was given by M. Tominaga in [13], using the Specht's ratio, in the following way

$$S\left(\frac{a}{b}\right) a^{\lambda} b^{1-\lambda} \geq \lambda a + (1-\lambda)b, \tag{1.3}$$

for positive real numbers a, b and $\lambda \in [0,1]$, where the Specht's ratio [3, 6, 12], was defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}},\ (h \neq 1)$$

for a positive real number h.

In [5], S. Furuichi improves inequality (1.1) thus

$$\lambda a + (1-\lambda)b \geq S\left(\left(\frac{a}{b}\right)^{r}\right) a^{\lambda} b^{1-\lambda} \tag{1.4}$$

where $= \min\{\lambda, 1-\lambda\}$ and the function S was given above.

H. Kober proved in [8] a general result related to an improvement of the inequality between arithmetic and geometric means, which for $n = 2$ implies the inequality

$$r(\sqrt{a}-\sqrt{b})^2 \leq \lambda a + (1-\lambda)b - a^{\lambda} b^{1-\lambda} \leq (1-r)(\sqrt{a}-\sqrt{b})^2, \tag{1.5}$$

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where a, b are the positive real numbers, $\lambda \in [0,1]$ and $r = \min\{\lambda, 1-\lambda\}$.

A generalization of inequality (1.5) can be found in a paper of J. M. Aldaz [1].
In [9], we present other improvement of the Young inequality and a reverse inequality as follows

$$a^{\lambda}b^{1-\lambda}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2r} \le \lambda a + (1-\lambda)b \le a^{\lambda}b^{1-\lambda}\left(\frac{a+b}{2\sqrt{ab}}\right)^{2(1-r)}, \tag{1.6}$$

for the positive real numbers a, b and $\lambda \in (0,1)$ where $r = \min\{\lambda, 1-\lambda\}$.

Inequality (1.6) was obtained by the inequality

$$\begin{aligned} 2p_{\min}\left(\frac{f(x_1)+f(x_2)}{2} - f\left(\frac{x_1+x_2}{2}\right)\right) &\le p_1 f(x_1) + p_2 f(x_2) \\ -f(p_1x_1+p_2x_2) &\le 2p_{\max}\left(\frac{f(x_1)+f(x_2)}{2} - f\left(\frac{x_1+x_2}{2}\right)\right), \end{aligned} \tag{1.7}$$

where $p_1 + p_2 = 1,\ p_{\min} = \min\{p_1, p_2\},\ p_{\max} = \max\{p_1, p_2\},\ x_1, x_2 > 0$, and f is a convex function, given by F. C. Mitroi [10], as a particular case of the Dragomir inequality [2].

2. Main results

Lemma 2.1. *For all x, y positive real numbers and $\lambda \in (0,1)$, we have the inequality*

$$2rE\left(x, y, \frac{1}{2}\right) \le E(x,y,\lambda) \le 2(1-r)E\left(x, y, \frac{1}{2}\right), \tag{2.8}$$

where

$$E(x,y,\lambda) = \lambda e^x + (1-\lambda)e^y - e^{\lambda x + (1-\lambda)y} - \frac{\lambda(1-\lambda)}{2}(x-y)^2$$

and $r = \min\{\lambda, 1-\lambda\}$.

Proof. We consider the function

$$f(t) = e^t - 1 - t - \frac{t^2}{2},\ (t > 0).$$

Since $f''(t) = e^t - 1 > 0$, for $t > 0$, it follows that f is a convex function.
Applying inequality (1.7) for $p_1 = \lambda > 0,\ x_1 = x,\ x_2 = y$ and for the function f, implies inequality (2.8). □

Theorem 2.1. *For $a, b \ge 1$ and $\lambda \in (0,1)$, we have*

$$\begin{aligned} r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)\log^2\left(\frac{a}{b}\right) &\le \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \\ &\le (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)\log^2\left(\frac{a}{b}\right), \end{aligned} \tag{2.9}$$

where $r = \min\{\lambda, 1-\lambda\},\ A(\lambda) = \dfrac{\lambda(1-\lambda)}{2} - \dfrac{r}{4}$ and $B(\lambda) = \dfrac{\lambda(1-\lambda)}{2} - \dfrac{1-r}{4}$.

Proof. Since $a, b \ge 1$, we use Lemma 2.1 for $x = \log a$ and $y = \log b$, which means that

$$E(\log a, \log b, \lambda) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - \frac{\lambda(1-\lambda)}{2}\log^2\frac{a}{b}.$$

and

$$E\left(\log a, \log b, \frac{1}{2}\right) = \frac{(\sqrt{a}-\sqrt{b})^2}{2} - \frac{1}{8}\log^2\frac{a}{b}.$$

Therefore, substituting the above relations in inequality (2.8), we deduce inequality of statement. □

Remark 2.1. Since $A(\lambda)=\dfrac{\lambda(1-\lambda)}{2}-\dfrac{r}{4}\geq 0$ and $B(\lambda)=\dfrac{\lambda(1-\lambda)}{2}-\dfrac{1-r}{4}\leq 0$, we obtain a refinement of the Kittaneh-Manasrah inequality and a refinement of inequality (1.5), in the following way:

$$\begin{aligned} r(\sqrt{a}-\sqrt{b})^2 &\leq r(\sqrt{a}-\sqrt{b})^2+A(\lambda)\log^2\left(\frac{a}{b}\right)\leq \lambda a+(1-\lambda)b-a^{\lambda}b^{1-\lambda}\\ &\leq (1-r)(\sqrt{a}-\sqrt{b})^2+B(\lambda)\log^2\left(\frac{a}{b}\right)\leq (1-r)(\sqrt{a}-\sqrt{b})^2, \end{aligned} \tag{2.10}$$

where $a,b\geq 1$.

It is easy to see that inequality (2.10) established a stronger inequality then Young's inequality.

Theorem 2.2. *For $x>-1$ and $\lambda\in(0,1)$, we have the inequality*

$$r(\sqrt{x+1}-1)^2\leq \lambda x+1-(x+1)^{\lambda}\leq (1-r)(\sqrt{x+1}-1)^2, \tag{2.11}$$

where $r=\min\{\lambda,1-\lambda\},\ A(\lambda)=\dfrac{\lambda(1-\lambda)}{2}-\dfrac{r}{4}$ and $B(\lambda)=\dfrac{\lambda(1-\lambda)}{2}-\dfrac{1-r}{4}$.

Proof. Substituting $\dfrac{a}{b}=t$ in inequality (2.10) we deduce the following inequality

$$r(\sqrt{t}-1)^2\leq \lambda t+1-\lambda-t^{\lambda}\leq (1-r)(\sqrt{t}-1)^2. \tag{2.12}$$

By replacing t with $x+1$ in relation (2.12), we find the inequality of the statement. □

Remark 2.2. Because, we have

$$r(\sqrt{t}-1)^2\geq 0,$$

we obtain an improvement of Bernoulli's inequality, $(x+1)^{\lambda}\leq \lambda x+1$, for $\lambda\in(0,1)$, and we give also a reverse inequality for the Bernoulli inequality.

3. Applications

Let s be a non-zero real number. For a sequence of positive weights $p_i,\ i=1...n$, we can define weighted power means of the positive real numbers $a_1,...,a_n$ as

$$M_s(a,p)=\left(\frac{\sum\limits_{i=1}^{n}p_ia_i^s}{\sum\limits_{i=1}^{n}p_i}\right)^{1/s}. \tag{3.13}$$

We might assume that the weights are normalized so that $\sum\limits_{i=1}^{n}p_i=1$, thus relations (3.13) becomes

$$M_s(a,\overline{p})=\left(\sum_{i=1}^{n}p_ia_i^s\right)^{1/s}.$$

It is known that, if $q<s$, then

$$M_q(a,p)\leq M_s(a,p) \tag{3.14}$$

and the two means are equal if and only if $a_1=a_2=...=a_n$.

Application 3.1. There is the inequality

$$\frac{r[M_s(a,p)]^{q-s}}{\sum_{i=1}^{n}p_i}\cdot\sum_{i=1}^{n}p_i(a_i^{s/2}-[M_s(a,p)]^{s/2})^2$$

$$+\frac{A\left(\frac{q}{s}\right)[M_s(a,p)]^{q-s}}{\sum_{i=1}^{n}p_i}\sum_{i=1}^{n}p_i\log^2\frac{a_i^s}{[M_s(a,p)]^s}\le[M_s(a,p)]^q-[M_q(a,p)]^q$$

$$\le\frac{(1-r)[M_s(a,p)]^{q-s}}{\sum_{i=1}^{n}p_i}\cdot\sum_{i=1}^{n}p_i\left(a_i^{s/2}-[M_s(a,p)]^{s/2}\right)^2$$

$$+\frac{B\left(\frac{q}{s}\right)[M_s(a,p)]^{q-s}}{\sum_{i=1}^{n}p_i}\cdot\sum_{i=1}^{n}p_i\log^2\frac{a_i^s}{[M_s(a,p)]^s},\tag{3.15}$$

where $0<q\le s,\ a_i\ge 1,\ p_i>0$, for all $i\in\{1,...,n\},\ r=\min\left\{\frac{q}{s},1-\frac{q}{s}\right\},\ A\left(\frac{q}{s}\right)=\frac{q(s-q)}{2s^2}-\frac{r}{4}$ and $B\left(\frac{q}{s}\right)=\frac{q(s-q)}{2s^2}-\frac{1-r}{4}$.

Proof. For $q=s$, we obtain the equality in relation (3.15).
For $q<s$, we take $\lambda=\frac{q}{s}<1$ and $a=a_i^s$ and $b=[M_s(a,p)]^s$ in inequality (2.9), thus, we obtain

$$r\left(\frac{a_i^{s/2}}{[M_s(a,p)]^{s/2}}-1\right)^2+\frac{A\left(\frac{q}{s}\right)}{[M_s(a,p)]^s}\log^2\frac{a_i^s}{[M_s(a,p)]^s}\le\frac{qa_i^s}{s[M_s(a,p)]^s}+1-\frac{q}{s}-\frac{a_i^q}{[M_s(a,p)]^q}$$

$$\le(1-r)\left(\frac{a_i^{s/2}}{[M_s(a,p)]^{s/2}}-1\right)^2+\frac{B\left(\frac{q}{s}\right)}{[M_s(a,p)]^s}\log^2\frac{a_i^s}{[M_s(a,p)]^s}.\tag{3.16}$$

Multiplying by p_i in inequality (3.16) and making the sum for $i=1...n$, we deduce the inequality

$$r[M_s(a,p)]^{q-s}\cdot\sum_{i=1}^{n}p_i(a_i^{s/2}$$

$$-[M_s(a,p)]^{s/2})+A\left(\frac{q}{s}\right)[M_s(a,p)]^q\cdot\sum_{i=1}^{n}p_i\log^2\frac{a_i^s}{[M_s(a,p)]^s}$$

$$\le\left(\sum_{i=1}^{n}p_i\right)([M_s(a,p)]^q-[M_q(a,p)]^q)$$

$$\le(1-r)[M_s(a,p)]^{q-s}\cdot\sum_{i=1}^{n}p_i(a_i^{s/2}-[M_s(a,p)]^{s/2})^2$$

$$+B\left(\frac{q}{s}\right)[M_s(a,p)]^{q-s}\cdot\sum_{i=1}^{n}p_i\log^2\frac{a_i^s}{[M_s(a,p)]^s}.$$

Dividing by $\sum_{i=1}^{n} p_i$, we have the inequality desired. □

Remark 3.3. From relation (3.15), we obtain an improvement of inequality (3.14) in case $0 < q \leq s$. For $p_i = \frac{1}{n}$ with $i = 1...n$ in inequality (3.14), we find the following inequality

$$\left(\frac{\sum_{i=1}^{n} a_i^q}{n}\right)^s \leq \left(\frac{\sum_{i=1}^{n} a_i^s}{n}\right)^q, \tag{3.17}$$

where $0 < q \leq s$.

Application 3.2. For $0 < a, b \leq 1$ and $\lambda \in (0, 1)$, we have

$$r(\sqrt{a} - \sqrt{b})^2 + A(\lambda) ab \log^2\left(\frac{a}{b}\right) \leq$$

$$\leq \lambda a + (1 - \lambda)b - a^\lambda b^{1-\lambda} \leq (1 - r)(\sqrt{a} - \sqrt{b})^2 + B(\lambda) ab \log^2\left(\frac{a}{b}\right), \tag{3.18}$$

where $r, A(\lambda), B(\lambda)$ are given in Theorem 2.1.

Proof. Since $0 < a, b \leq 1$, we deduce $\frac{1}{a}, \frac{1}{b} \geq 1$.
Applying Theorem 2.1 for $a \to \frac{1}{a},\ b \to \frac{1}{b}$, we obtain the relation

$$r(\sqrt{a} - \sqrt{b})^2 + A(\lambda) ab \log^2\left(\frac{a}{b}\right) \leq \lambda b + (1 - \lambda)a - b^\lambda a^{1-\lambda}$$

$$\leq (1 - r)(\sqrt{a} - \sqrt{b})^2 + B(\lambda) ab \log^2\left(\frac{a}{b}\right). \tag{3.19}$$

For $\lambda \to 1 - \lambda$ and taking into account that $A(1 - \lambda) = A(\lambda)$ and $B(1 - \lambda) = B(\lambda)$ in relation (3.19) we have relation (3.18). □

In [11], the beta function B is the real function of two variables defined by the formula

$$B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt,\ for\ x, y > 0.$$

Application 3.3. For $x, y, z \geq 1$ and $\lambda \in (0, 1)$, we have

$$r\left[B(x, z) + B(y, z) - 2B\left(\frac{x + y}{2}, z\right)\right] + A(\lambda)(x - y)^2 \int_0^1 t^{x+y-2}(1 - t)^{2z-2} \log^2 t dt$$

$$\leq \lambda B(x, z) + (1 - \lambda)B(y, z) - B(\lambda x + (1 - \lambda)y, z)$$

$$\leq (1 - r)\left[B(x, z) + B(y, z) - 2B\left(\frac{x + y}{2}, z\right)\right]$$

$$+B(\lambda)(x - y)^2 \int_0^1 t^{x+y-2}(1 - t)^{2z-2} \log^2 t dt, \tag{3.20}$$

where $r, A(\lambda), B(\lambda)$ are given in Theorem 2.1.

Proof. For $a = t^{x-1}(1 - t)^{z-1}$ and $b = t^{y-1}(1 - t)^{z-1}$ we have $0 < a, b < 1$, when $x, y, z \geq 1$ and $t \in (0, 1)$.
Therefore, we use Application 3.2 and we obtain an inequality which by integra-ting from 0 to 1 implies relation (3.20). □

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"DIMITRIE CANTEMIR" UNIVERSITY
BISERICII ROMÂNE 107
500026 BRAŞOV, ROMÂNIA
E-mail address: minculeten@yahoo.com