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Topological indices of the double odd graph $2O_k$

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ABSTRACT. In this paper we use transitivity property of the automorphism group of the double odd graph to calculate its Wiener, Szeged and PI indices.

1. INTRODUCTION

All graphs in this paper are simple and connected. In general a graph is denoted by $G = (V, E)$, where V is the set of vertices of G and E is the set of edges of G. If $z, u, v \in V$, and $e = uv \in E$, then the edge connecting u to v is denoted by $uv \in E$. Also all graphs considered in this paper are finite in a sense that both V and E are finite sets. For vertices u and v in V, a path from u to v is a sequence of vertices $u = u_0, u_1, ..., u_n = v$ such that $u_iu_{i+1} \in$ E where $0 \leq i \leq n-1$. In this case n is called the length of the path form u to v. The length of the shortest path from u to v is called the distance between u and v and is denoted by $d(u, v)$ and $d(z, e) := min\{d(u, z),\ d(v, z)\}$. The Wiener index of the graph G is denoted by $W(G)$ and is defined by:

$$W(G) = \sum_{\{u,v\} \subseteq V} d(u, v).$$

If the sum of distances from a vertex v in V is denoted by $d(v)$, i. e.

$$d(v) = \sum_{x \in V} d(v, x),$$

then

$$W(G) = (1/2) \sum_{v \in V} d(v).$$

The Wiener index for the first time was proposed in [14] in connection with the boiling points of chemical substances. The definition of the Wiener index in terms of the distances between vertices of a graph is give by Hosoya in [9]. Because of the above chemical fact about the Wiener index and also the fact that it is an invariant of the graph, i. e. it is invariant under the automorphism group of the graph, various methods have been developed to calculate it, for example one can refer to [4] and [7].
Apart from the Wiener index, there are numerous indices associated to a graph which are invariant under the automorphism group of the graph. But the next topological index that we are interested in is called the Szeged index and is define as follows. Let $G = (V, E)$ be a simple connected graph and $e = uv$ be an edge in E. By $n_u(e|G)$ we mean the number of vertices in V lying closer to u than v. The quantity $n_v(e|G)$ is defined similarly. Therefore if we define the sets

$$N_u(e|G) = \{w \in V \mid d(w, u) < d(w, v)\},$$

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and

$$N_v(e|G) = \{w \in V \mid d(w,v) < d(w,u)\},$$

then we set

$$n_u(e|G) = |N_u(e|G)|,$$

and

$$n_v(e|G) = |N_v(e|G)|.$$

The Szeged index of the graph $G = (V, E)$ is defined by the formula

$$Sz(G) = \sum_{e=uv \in E} n_u(e|G) n_v(e|G).$$

Because of the importance of the Szeged index its calculation has been studied by several authors. To mention a few, one can refer to $[5, 8, 10]$. A different method of calculating the topological indices, based on the properties of the automorphism group of the graph, was initiated in [15]. This method was extended and applied to some graphs in [1] and [2]. Recently this method is used in [16] to compute the Szeged index of a symmetric graph.

Since the Szeged index takes into account how the vertices of the graph G are distributed, it is natural to define an index that takes into account the distribution of the edges of G. The Padmakar-Ivan PI-index, $[11, 12]$ is an additive index which takes into account the distribution of the edges of the graph and therefore complements the Szeged index in a certain sense. The next topological index that we are interested in is called the PI-index and is defined as follows.Let $G = (V, E)$ be a simple connected graph and $e = uv$ be an edge in E. By $n_{eu}(e|G)$ we mean the number of edges in E lying closer to u than v. The quantity $n_{ev}(e|G)$ is defined similarly. Therefore if we define the sets

$$N_{eu}(e|G) = \{f \in E \mid d(f,u) < d(f,v)\},$$

and

$$N_{ev}(e|G) = \{f \in E \mid d(f,v) < d(f,u)\},$$

then we define

$$n_{eu}(e|G) = |N_{eu}(e|G)|,$$

and

$$n_{ev}(e|G) = |N_{ev}(e|G)|.$$

The PI-index of the graph $G = (V, E)$ is defined by the formula

$$PI(G) = \sum_{e=uv \in E} \Big(n_{eu}(e|G) + n_{ev}(e|G)\Big).$$

In this paper our aim is to provide some concepts and proofs concerning this method which uses group theory For materials from the theory of groups and graph theory one can see [3] and [6]. We also use this method to calculate the Wiener, the Szeged and PI-index of the double odd graph.

2. Concepts and results

In this section we will use some of definitions and theorems in [13] to calculate the Wiener, the Szeged and PI-index of graphs.

Definition 2.1. Let G be a group which acts on a set X. Let us denote the action of $\sigma \in G$ on $x \in X$ by x^σ. Then G is said to act transitively on X if for every $x, y \in X$ there is $\sigma \in G$ such that $x^\sigma = y$.

Definition 2.2. Let $G = (V, E)$ be a graph. An automorphism σ of G is a one-to-one mapping from V onto V which preserves adjacency, i. e. $e = uv$ is an edge of G if and only if $e^\sigma = u^\sigma v^\sigma$ is also an edge of G.

The set of all the automorphisms of the graph G is a group under the usual composition of mappings. This group is denoted by $\mathbb{A}ut(G)$ and is a subgroup of the symmetric group on X.
From Definition 2.2 it is clear that $\mathbb{A}ut(G)$ acts on the set V of vertices of G. This action induces an action on the set E of edges of G. In fact if $e = uv$ is an edge of G and $\sigma \in \mathbb{A}ut(G)$ then $e^\sigma := u^\sigma v^\sigma$ is an edge of G and this is a well-defined action $\mathbb{A}ut(G)$ on E.

Definition 2.3. Let $G = (V, E)$ be a graph. G is called vertex-transitive if $\mathbb{A}ut(G)$ acts transitively on the set X of vertices of G. If $\mathbb{A}ut(G)$ acts transitively on the set E of edges of G, then G is called an edge-transitive graph.

Theorem 2.1. *Let $G = (V, E)$ be a simple vertex-transitive graph and let $v \in V$ be a fixed vertex of G. Then*

$$W(G) = (1/2)\ |V|\ d(v),$$

where

$$d(v) = \sum_{x \in V} d(v, x).$$

Theorem 2.2. *Let $G = (V, E)$ be a simple edge-transitive graph and let $e = uv$ be a fixed edge of G. Then the Szeged index of G is as follows:*

$$Sz(G) = |E| n_u(e|G) n_v(e|G).$$

Theorem 2.3. *Let $G = (V, E)$ be a simple edge-transitive graph and let $e = uv$ be a fixed edge of G. Then the PI-index of G is as follows:*

$$PI(G) = |E|\Big(n_{eu}(e|G) + n_{ev}(e|G)\Big).$$

3. Computing the Wiener, the Szeged and the PI-index of the double odd graph

Definition 3.4. For a positive integer $k \geq 2$, let X be any set of cardinality $2k - 1$. The collection of all $(k-1)$-subsets and k-subsets of X which are denoted by X_{k-1} and X_k respectively. The double odd graph $2O_k$ has $V = X_k \cup X_{k-1}$ as its vertex set, and vertex α of X_{k-1} is adjacent to vertex β of X_k if and only if $\alpha \subset \beta$.

Therefore $2O_k$ has $2\binom{2k-1}{k}$ vertices, it is regular of degree k and it is a bipartite graph. The number of edges of $2O_k$ is $k\binom{2k-1}{k}$. If σ is a permutation of Ω and $A \subseteq \Omega$, then A^σ is defined by: $A^\sigma = \{a^\sigma | a \in A\}$ which is again a subset of Ω of cardinality $|A|$. Therefore each permutation of Ω induces a permutation on the set of vertices of $2O_k$. If AB is an edge of $2O_k$, then A and B are subset of Ω with cardinality $k-1$ and k respectively, where $A \subset B$ and for any permutation σ of Ω we have $A^\sigma \subset B^\sigma$, hence $A^\sigma B^\sigma$ is an edge of $2O_k$, which proves that σ is an element of $\mathbb{A}ut(2O_k)$. Therefore we have proved the following theorem:

Theorem 3.4. *The automorphism group of the double odd group $2O_k$ contains a subgroup isomorphic to the symmetric group on $2k-1$ letters.*

From the above fact we can show vertex and edge transitivity of double odd graphs.

Lemma 3.1. *the double odd graph is both vertex and edge transitive.*

Proof. Let Ω be a set of size $2k-1$. Without loss of generality we may assume $\Omega = \{1,2,...,2k-1\}$. Let the double odd graph be defined on Ω. Consider two distinct vertices A and B of $2O_k$. We may assume $A=\{1,2,...,k-1\}$ (or$\{1,2,...,k-1,k\}$), $B=\{1',2',...,(k-1)'\}$ (or$\{1',2',...,(k-1)',k'\}$). Then we set $\Omega-A=\{k,...,n\}$ $(or\ \{k+1,...,n\})$ and $\Omega-B=\{k',...,n'\}$ $(or\ \{(k+1)',...,n'\})$ and both are subsets of Ω. Then $\pi:\Omega\rightarrow\Omega$ defined by $i\rightarrow i'$ is an element of the symmetric group S_{2k-1} which induces an element of $\mathbb{A}ut(2O_k)$ and $A^\pi=B$. This proves that $2O_k$ is vertex-transitive.
Now assume AB and CD are distinct edges of $2O_k$. To prove edge-transitivity of $2O_k$ it is enough to show that there is a permutation π on Ω such that $A^\pi=C$ and $B^\pi=D$. Without loss of generality we may assume that $A=\{1,2,...,k-2,k-1\}$, $B=\{1,2,...,k-1,k\}$, $C=\{1',2',...,(k-1)'\}$, $D=\{1',2',...,(k-1)',k'\}$. Then we set $\Omega-(A\cup B)=\{K+1,...,n\}$ and $\Omega-(C\cup D)=\{(k+1)',...,n'\}$ and both subsets Ω. Now the permutation $\pi:\Omega\rightarrow\Omega$ by defined $i\rightarrow i'$ has the required property and the lemma is proved. □

The following tables are used in further results. Note that $A\Delta B$ denotes the symmetric difference of the sets A and B, i. e. $A\Delta B=(A\cup B)-(A\cap B)$. By definition $2O_k$ the results in Table 1 are obvious,

Table 1
Distance $d(A,B)$ and corresponding $|A\Delta B|$

$d(A,B)$	0	1	2	3	4	5	...	2k-1
$\|A\Delta B\|$	0	1	2	3	4	5	...	2k-1

and moreover by the properties of bipartite graphs the results in Tables 2, 3, 4 are obvious. Therefore we have
(i) If $A\in X_{k-1}$ and $B\in X_k$ then

Table 2
Distance $d(A,B)$ and corresponding $|A\cap B|$

$d(A,B)$	1	3	5	...	2k-1
$\|A\cap B\|$	k-1	k-2	k-3	...	0

(ii) If $A,B\in X_k$ then

Table 3
Distance $d(A,B)$ and corresponding $|A\cap B|$

$d(A,B)$	0	2	4	...	2k-2
$\|A\cap B\|$	k	k-1	k-2	...	1

(iii) If $A,B\in X_{k-1}$ then

Table 4
Distance $d(A,B)$ and corresponding $|A\cap B|$

$d(A,B)$	0	2	4	...	2k-2
$\|A\cap B\|$	k-1	k-2	k-3	...	0

Theorem 3.5. *The Wiener index of $2O_k$, $k\geq 2$, is:*

$$W(2O_k)=\binom{2k-1}{k}\left(\sum_{i=1}^{k}(2i-1)\binom{k}{k-i}\binom{k-1}{i-1}+\sum_{i=1}^{k-1}2i\binom{k}{k-i}\binom{k-1}{i}\right)$$

Proof. By Theorem 3.4, $2O_k$ is vertex-transitive and by Theorem 2.1:

$$W(2O_k) = \binom{2k-1}{k} d(A)$$

where A is a fixed vertex of $2O_k$ and $d(A) = \sum_B d(A, B)$, where B is a subset of Ω with cardinality k or $k-1$. By Tables $1, 2, 3, 4$ for any vertices like $u \in V$ the number of vertices like v such that $d(u, v) = i$, $0 \leq i \leq 2k-1$ is calculated as follows:
if $d(u, v) = 0$,then the number of choices for v is 1, if $d(u, v) = 1$ then the number of choices for v is $\binom{k}{k-1}$, if $d(u, v) = 2$ then the number of choices for v is $\binom{k}{k-1}\binom{k-1}{1}$ and continue this method until $d(u, v) = 2k-1$ that in this case the number of choices for v is $\binom{k}{k-k}\binom{k-1}{k-1}$. Therefore we have

$$W(2O_k) = \binom{2k-1}{k}\Bigg(\binom{k}{k-1} + 2\binom{k}{k-1}\binom{k-1}{1} + 3\binom{k}{k-2}\binom{k-1}{1} +$$

$$... + (2k-2)\binom{k}{k-(k-1)}\binom{k-1}{k-1} + (2k-1)\binom{k}{k-k}\binom{k-1}{k-1}\Bigg),$$

hence

$$W(2O_k) = \binom{2k-1}{k}\Bigg(\sum_{i=1}^{k}(2i-1)\binom{k}{k-i}\binom{k-1}{i-1} + \sum_{i=1}^{k-1} 2i\binom{k}{k-i}\binom{k-1}{i}\Bigg).$$

□

Lemma 3.2. *Let $e = uv \in E(2O_k)$. To calculate $n_u(e|2O_k)$ it is enough to calculate the number of vertices like z in V such that $d(u, z) \leq 2k-2$ and $d(u, z) < d(v, z)$.*

Proof. For vertices like u, v, z such that $u \neq v$ we have $2k$ cases:
(1) If $d(u, z) = 0$, then $z = u$, therefore $z \in N_u(e|O_k)$,
(2) If $d(u, z) = 1$, then $d(v, z) = 0$, or $2, \ldots, 2k-2$, now if $d(v, z) = 1$, then $z \notin N_u(e|O_k)$ otherwise $z \in N_u(e|O_k)$,
...
(2k-1) If $d(u, z) = 2k-2$, then $d(v, z) = 1$ or $3, \ldots, 2k-1$, now if $d(v, z) = 1$ or $3, ..., 2k-3$, then $z \notin N_u(e|O_k)$ otherwise $z \in N_u(e|O_k)$,
(2k) If $d(u, z) = 2k-1$, then $d(v, z) = 0$ or $2, \ldots, 2k-2$, therefore $z \notin N_u(e|O_k)$.

□

Theorem 3.6. *The Szeged index of $2O_k$, $k \geq 2$, is:*

$$Sz(2O_k) = k\binom{2k-1}{k}\Big(\sum_{i=1}^{2k-2} E_i\Big)^2$$

where

$$E_0 = 1 \quad , E_1 = \binom{k}{k-1} - E_0 \quad ,$$

$$E_i = \begin{cases} \binom{k}{k-i}\binom{k-1}{i-1} - E_{i-1} & \text{if } i \text{ is odd,} \\ \binom{k-1}{i}\binom{k}{k-i} - E_{i-1} & \text{if } i \geq 2 \ , i \text{ is even.} \end{cases}$$

Proof. Since by Theorem 3.4, $2O_k$ is edge-transitive, we can use Theorem 2.2 to write

$$Sz(2O_k) = k\binom{2k-1}{k} n_u(e|2O_k) n_v(e|2O_k).$$

where $e = uv$ is a fixed edge of $2O_k$. Since $2O_k$ is a symmetric graph hence $n_u(e|2O_k) = n_v(e|2O_k)$, therefore

$$Sz(2O_k) = k\binom{2k-1}{k}\Big(n_u(e|2O_k)\Big)^2.$$

We proceed to calculate $n_u(e|2O_k)$. Without loss of generality we can assume $u \in X_{k-1}$ and $v \in X_k$. We define E_i, $0 \le i \le 2k-2$ to be the number of vertices like x in V such that $d(u,x) = i$ and $d(u,x) < d(v,x)$. Now we calculate E_i where $0 \le i \le k-2$. By Table 4 we have $E_0 = 1$ because $d(u,x) = 0$ if and only if $|u \cap x| = k-1$, $E_1 = \binom{k}{k-1} - E_0$ where by Table 2, $\binom{k}{k-1}$ is the number of choices for vertices like $y \in V$ such that $d(u,y) = 1$ and E_0 is the number of choices for vertices in V like w such that $d(w,v) = 0$ so these vertices must decrease, $E_2 = \binom{k-1}{k-2}\binom{k}{1} - E_1$ where by Table 4, $\binom{k-1}{k-2}\binom{k}{1}$ is the number of choices for vertices like $a \in V$ such that $d(u,a) = 2$ and E_1 is the number of vertices like r in V such that $d(v,r) = 1$ so these vertices must decrease, hence by Lemma 3.2 we must continue this method until $d(u,x) = 2k-2$. Then we have

$$E_0 = 1 \quad, E_1 = \binom{k}{k-1} - E_0 \quad and$$

$$E_i = \begin{cases} \binom{k}{k-i}\binom{k-1}{i-1} - E_{i-1} & \text{if } i \text{ is odd,} \\ \binom{k-1}{i}\binom{k}{k-i} - E_{i-1} & \text{if } i \ge 2,\ i \text{ is even.} \end{cases}$$

and therefore we have

$$Sz(2O_k) = k\binom{2k-1}{k}\Big(\sum_{i=1}^{2k-2} E_i\Big)^2.$$

□

Lemma 3.3. *Let G be a connected graph, then we have*

$$PI(G) = |E(G)|^2 - \sum_{e \in E(G)} N(e)$$

where $e = uv$ is a fixed edge of G and $N(e)$ is the number of edges equidistance from u and v.

Proof. By definition of $PI(G)$ we have

$$PI(G) = \sum_{e \in E(G)} \Big(n_{eu}(e|G) + n_{ev}(e|G)\Big)$$

Since $E(G) = n_{eu}(e|G) + n_{ev}(e|G) + N(e)$, hence

$$E(G) - N(e) = n_{eu}(e|G) + n_{ev}(e|G)$$

then we have

$$PI(G) = \sum_{e \in E(G)} \Big(|E(G) - N(e)\Big) = |E(G)|^2 - \sum_{e \in E(G)} N(e).$$

□

Theorem 3.7. *The PI-index of $2O_k$, $k \geq 2$, is*

$$PI(2O_k) = k^2 \binom{2k-1}{k}^2 - k\binom{2k-1}{k}\big(E_0 + E_2 + ... + E_{2k-2}\big)$$

where

$$E_0 = 1, \quad E_1 = \binom{k}{k-1} - E_0 \quad \textit{and}$$

$$E_i = \begin{cases} \binom{k}{k-i}\binom{k-1}{i-1} - E_{i-1} & \textit{if } \ i \ \textit{ is odd,} \\ \binom{k-1}{i}\binom{k}{k-i} - E_{i-1} & \textit{if } \ i \geq 2, \ \ i \ \textit{ is even.} \end{cases}$$

Proof. Since by Theorem 3.4, $2O_k$ is edge-transitive, we can use Theorem 2.3 and Lemma 3.3 to write

$$PI(2O_k) = k^2 \binom{2k-1}{k}^2 - k\binom{2k-1}{k}N(e)$$

where $e = uv$ is a fixed edge of $2O_k$. First we calculate $N(e)$. In fact it is obvious that by the properties of bipartite graphs we must calculate the number of vertices like w in E such that $d(u,w) = d(v,w) = 2i$, $0 \leq i \leq k-1$. Therefore we can define E_i, $0 \leq i \leq 2k-2$, in the same manner as in the proof of Theorem 3.6. Then we have $E_0 = 1$, $E_1 = \binom{k}{k-1} - E_0$ and

$$E_i = \begin{cases} \binom{k}{k-i}\binom{k-1}{i-1} - E_{i-1} & \text{if } \ i \ \text{ is odd,} \\ \binom{k-1}{i}\binom{k}{k-i} - E_{i-1} & \text{if } \ i \geq 2, \ \ i \ \text{ is even.} \end{cases}$$

Therefore we have

$$PI(2O_k) = k^2 \binom{2k-1}{k}^2 - k\binom{2k-1}{k}\big(E_0 + E_2 + ... + E_{2k-2}\big).$$

□

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