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Solvability of a nonlinear general third order four point eigenvalue problem on time scales

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ABSTRACT. We consider the four point boundary value problem for third order nonlinear differential equation on time scales

$$y^{\Delta^3}(t) + \lambda f(t, y, y^{\Delta}, y^{\Delta^2}) = 0,\ t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}$$

subject to the boundary conditions

$$y(t_1) = 0,\ y(t_2) = 0,\ \beta y(t_3) - \alpha y(\sigma^3(t_4)) = 0$$

where $t_1 \le t_2 \le t_3 \le \sigma^3(t_4), \alpha > 0, \beta > 0$. Values of the parameter λ are determined for which the boundary value problem has a positive solution by utilizing a fixed point theorem on cone.

1. INTRODUCTION

The main focus of this paper is to determine the intervals of eigenvalues, λ, for which there exist positive solutions with respect to a cone, of third order nonlinear differential equation on time scale with four point boundary conditions. The study of the existence of positive solutions of boundary value problems (BVPs) on optimal intervals for higher order differential equations on time scales has gained prominence since it arises in many applications. One goal as the result of Hilger's [18] initial paper introducing time scales has been the unification of the continuous and discrete calculus, and then the extension of those results to dynamic equations on time scales. For an excellent introduction to the overall area of dynamic equations on time scales, we refer to text book by Bohner and Peterson [7].

One particular area receiving current attention is the question of obtaining optimal eigenvalue intervals of boundary value problems for ordinary differential equations, as well as for finite difference equations. Many of these works have used Krasnosel'skii fixed point theorems [20] to obtain intervals based on a positive solutions inside a cone. A few papers along these lines are Agarwal, Bohner, and Wong [2], Anderson and Davis [6], Davis, Henderson, Prasad, and Yin [10], Eloe and Henderson [13], Erbe and Wang [15], Erbe and Tang [14], Erbe and Peterson [17]. Naturally many of these methods carry over when determining eigenvalue intervals of boundary value problems for dynamic equations on time scales; see Agarwal, Bohner, and Wong [1], Anderson [3, 4, 5], Chyan and Henderson [9], Davis, Henderson, Prasad and Yin [11], and Erbe and Peterson [16], Prasad, Nageswara Rao and Murali [21].
We introduce the delta derivative y^{Δ} for the function y defined on $\mathbb{T}$,
$y^{\Delta} = y'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and $y^{\Delta} = \Delta y$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

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In this paper, we determine the eigenvalue intervals for which there exists positive solution for four point boundary value problem on time scales

$$y^{\Delta^3}(t) + \lambda f(t, y, y^{\Delta}, y^{\Delta^2}) = 0,\ t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}, \tag{1.1}$$

$$y(t_1) = 0,\ y(t_2) = 0,\ \beta y(t_3) - \alpha y(\sigma^3(t_4)) = 0. \tag{1.2}$$

We use the following notation for the convenience,

$$\begin{aligned} A &= \beta t_3 - \alpha\sigma^3(t_4),\ B = \beta t_3^2 - \alpha(\sigma^3(t_4))^2, \\ k_1 &= \frac{t_1^2(\beta-\alpha) - B}{2(A - t_2(\beta-\alpha))},\ k_2 = \frac{t_1^2(\beta-\alpha) - B}{2(t_1(\beta-\alpha) - A)} \\ \text{and } D &= A(t_1^2 - t_2^2) - (\beta-\alpha)(t_1^2 t_2 - t_1 t_2^2) + B(t_2 - t_1). \end{aligned}$$

We make the following assumptions throughout:

(A1) $f : [t_1, \sigma^3(t_4)]_{\mathbb{T}} \times \mathbb{R}^{+^3} \to \mathbb{R}^+$ is continuous with respect to y, where $\mathbb{R}^+$ is the set of positive real numbers,
(A2) $\beta > \alpha > 0$ and $t_1 \le t_2 \le t_3 \le \sigma^3(t_4)$,
(A3) $t_2 < \beta t_3 + \alpha\sigma^3(t_4)$,
(A4) the point $t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}$ is not left dense and right scattered at the same time.

We define the positive extended real numbers $f_0, f^0, f_\infty, f^\infty$ by

$$f_0 = \lim_{(y,y^{\Delta},y^{\Delta^2})\to(0^+,0^+,0^+)} \inf_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y},$$

$$f^0 = \lim_{(y,y^{\Delta},y^{\Delta^2})\to(0^+,0^+,0^+)} \sup_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y},$$

$$f_\infty = \lim_{(y,y^{\Delta},y^{\Delta^2})\to(\infty,\infty,\infty)} \inf_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y},$$

$$f^\infty = \lim_{(y,y^{\Delta},y^{\Delta^2})\to(\infty,\infty,\infty)} \sup_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y},$$

and assume that they will exist.

This paper is organized as follows. In Section 2, we estimate the bounds for the Green's function and established related lemma. In Section 3, we establish a criteria to determine eigenvalue intervals for which there exist at least one positive solution of the BVP (1.1)-(1.2) by using Krasnosel'skii fixed point theorem. Finally, as an application, we give examples to demonstrate our results.

2. Green's function and bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to the BVP (1.1)-(1.2) in three different intervals in twelve different cases and we estimate the bounds for the Green's function.

Let $G(t,s)$ be the Green's function for the BVP $-y^{\Delta^3}(t) = 0$ satisfying (1.2). After computation, the Green's function $G(t,s)$ can be obtained as

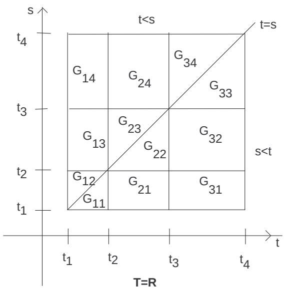
$$G(t,s) = \begin{cases} \underset{t\in[t_1,t_2]_{\mathbb{T}}}{G(t,s)} = \begin{cases} G_{11}(t,s), & \mathrm{t_1 \le \sigma(s) < t \le t_2 < t_3 < \sigma^3(t_4)}, \\ G_{12}(t,s), & \mathrm{t_1 \le t < s \le t_2 < t_3 < \sigma^3(t_4)}, \\ G_{13}(t,s), & \mathrm{t_1 \le t \le t_2 < s < t_3 < \sigma^3(t_4)}, \\ G_{14}(t,s), & \mathrm{t_1 \le t \le t_2 < t_3 < s < \sigma^3(t_4)}, \end{cases} \\ \underset{t\in[t_2,t_3]_{\mathbb{T}}}{G(t,s)} = \begin{cases} G_{21}(t,s), & \mathrm{t_1 < \sigma(s) < t_2 \le t \le t_3 < \sigma^3(t_4)}, \\ G_{22}(t,s), & \mathrm{t_1 < t_2 \le \sigma(s) < t < t_3 < \sigma^3(t_4)}, \\ G_{23}(t,s), & \mathrm{t_1 < t_2 \le t < s \le t_3 < \sigma^3(t_4)}, \\ G_{24}(t,s), & \mathrm{t_1 < t_2 \le t \le t_3 < s < \sigma^3(t_4)}, \end{cases} \\ \underset{t\in[t_3,\sigma^3(t_4)]_{\mathbb{T}}}{G(t,s)} = \begin{cases} G_{31}(t,s), & \mathrm{t_1 < \sigma(s) < t_2 < t_3 \le t \le \sigma^3(t_4)}, \\ G_{32}(t,s), & \mathrm{t_1 < t_2 < \sigma(s) < t_3 \le t \le \sigma^3(t_4)}, \\ G_{33}(t,s), & \mathrm{t_1 < t_2 < t_3 \le \sigma(s) < t \le \sigma^3(t_4)}, \\ G_{34}(t,s), & \mathrm{t_1 < t_2 < t_3 \le t < s \le \sigma^3(t_4)}, \end{cases} \end{cases} \tag{2.3}$$

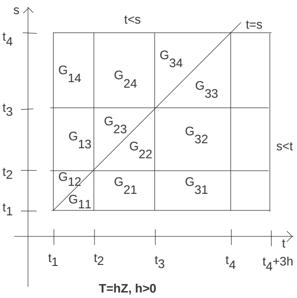
where

$$\begin{aligned}
G_{11}(t,s) &= \frac{1}{2D}[(At_2^2 - Bt_2) - t(t_2^2(\beta-\alpha) - B) - t^2(A - t_2(\beta-\alpha))][t_1 - \sigma(s)]^2, \\
G_{12}(t,s) &= \frac{1}{2D}[(At_1^2 - Bt_1) + t(B - t_1^2(\beta-\alpha)) + t^2(t_1(\beta-\alpha) - A)][t_2 - \sigma(s)]^2 \\
&\quad + \frac{1}{2D}[(t_1t_2^2 - t_1^2t_2) + t(t_1^2 - t_2^2) + t^2(t_2 - t_1)][(\beta-\alpha)\sigma(s)\sigma^2(s) \\
&\qquad - A(\sigma(s) + \sigma^2(s)) + B], \\
G_{13}(t,s) &= \frac{1}{2D}[(t_1t_2^2 - t_1^2t_2) + t(t_1^2 - t_2^2) + t^2(t_2 - t_1)][(\beta-\alpha)\sigma(s)\sigma^2(s) \\
&\qquad - A(\sigma(s) + \sigma^2(s)) + B], \\
G_{14}(t,s) &= \frac{1}{2D}[-(t_1t_2^2 - t_1^2t_2) - t(t_1^2 - t_2^2) - t^2(t_2 - t_1)][\alpha(\sigma^3(t_4) - \sigma(s))^2], \\
G_{21}(t,s) &= \frac{1}{2D}[(At_2^2 - Bt_2) - t(t_2^2(\beta-\alpha) - B) - t^2(A - t_2(\beta-\alpha))][t_1 - \sigma(s)]^2, \\
G_{22}(t,s) &= \frac{1}{2D}[(At_2^2 - Bt_2) - t(t_2^2(\beta-\alpha) - B) - t^2(A - t_2(\beta-\alpha))][t_1 - \sigma(s)]^2 \\
&\quad + \frac{1}{2D}[-(At_1^2 - Bt_1) + t(t_1^2(\beta-\alpha) - B) + t^2(A - t_1(\beta-\alpha))][t_2 - \sigma(s)]^2, \\
G_{23}(t,s) &= \frac{1}{2D}[(t_1t_2^2 - t_1^2t_2) + t(t_1^2 - t_2^2) + t^2(t_2 - t_1)][(\beta-\alpha)\sigma(s)\sigma^2(s) \\
&\qquad - A(\sigma(s) + \sigma^2(s)) + B], \\
G_{24}(t,s) &= \frac{1}{2D}[-(t_1t_2^2 - t_1^2t_2) - t(t_1^2 - t_2^2) - t^2(t_2 - t_1)][\alpha(\sigma^3(t_4) - \sigma(s))^2], \\
G_{31}(t,s) &= \frac{1}{2D}[(At_2^2 - Bt_2) - t(t_2^2(\beta-\alpha) - B) - t^2(A - t_2(\beta-\alpha))][t_1 - \sigma(s)]^2, \\
G_{32}(t,s) &= \frac{1}{2D}[(At_2^2 - Bt_2) - t(t_2^2(\beta-\alpha) - B) - t^2(A - t_2(\beta-\alpha))][t_1 - \sigma(s)]^2 \\
&\quad + \frac{1}{2D}[-(At_1^2 - Bt_1) + t(t_1^2(\beta-\alpha) - B) + t^2(A - t_1(\beta-\alpha))][t_2 - \sigma(s)]^2,
\end{aligned}$$

$$G_{33}(t,s) = \frac{1}{2D}[(At_2^2 - Bt_2) - t(t_2^2(\beta-\alpha) - B) - t^2(A - t_2(\beta-\alpha))][t_1 - \sigma(s)]^2$$
$$+\frac{1}{2D}[-(At_1^2 - Bt_1) + t(t_1^2(\beta-\alpha) - B) + t^2(A - t_1(\beta-\alpha))][t_2 - \sigma(s)]^2$$
$$+\frac{1}{2D}[(t_1t_2^2 - t_1^2t_2) + t(t_1^2 - t_2^2) + t^2(t_2 - t_1)][\beta(t_3 - \sigma(s))^2],$$
$$G_{34}(t,s) = \frac{1}{2D}[-(t_1t_2^2 - t_1^2t_2) - t(t_1^2 - t_2^2) - t^2(t_2 - t_1)][\alpha(\sigma^3(t_4) - \sigma(s))^2].$$

The following graphs demonstrate the Green's function for the BVP (1.1)-(1.2) should be taken in the form of (2.3). Here $s \in [t_1, t_4]$.





Theorem 2.1. *Let $G(t,s)$ be the Green's function for the homogeneous problem $-y^{\Delta^3}(t) = 0$ satisfying the boundary conditions* (1.2). *Then the inequality below holds*

$$mG(\sigma(s),s) \le G(t,s) \le G(\sigma(s),s), \ \textit{for all } (t,s) \in [t_1, \sigma^3(t_4)]_{\mathbb{T}} \times [t_1, t_4], \tag{2.4}$$

where

$$0 < m = \min\left\{ \frac{G_{11}(k_1,s)}{G_{11}(\sigma(s),s)}, \frac{G_{13}(\frac{t_1}{2},s)}{G_{13}(\sigma(s),s)}, \frac{G_{14}(\frac{t_1}{2},s)}{G_{14}(\sigma(s),s)}, \frac{G_{12}(k_2,s)}{G_{12}(\sigma(s),s)}, \frac{G_{22}(k_2,s)}{G_{22}(\sigma(s),s)}, \frac{G_{33}(\frac{t_1}{2},s)}{G_{33}(\sigma(s),s)} \right\} < 1. \tag{2.5}$$

Proof. The Green's function $G(t,s)$ is given in (2.3) in twelve different cases. In each case we prove the inequality as in (2.4). Clearly

$$G(t,s) > 0 \text{ on } [t_1, \sigma^3(t_4)]_{\mathbb{T}} \times [t_1, t_4]. \tag{2.6}$$

Case (i). For $t_1 \le \sigma(s) < t \le t_2 < t_3 < \sigma^3(t_4)$

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{11}(t,s)}{G_{11}(\sigma(s),s)} = \frac{[(At_2^2 - Bt_2) - t(t_2^2(\beta-\alpha) - B) - t^2(A - t_2(\beta-\alpha))]}{[(At_2^2 - Bt_2) - \sigma(s)(t_2^2(\beta-\alpha) - B) - (\sigma(s))^2(A - t_2(\beta-\alpha))]}.$$

From $(A2)$ and $(A3)$ we have $G_{11}(t,s) \le G_{11}(\sigma(s),s)$. Therefore, $G(t,s) \le G(\sigma(s),s)$. And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{11}(t,s)}{G_{11}(\sigma(s),s)} \ge \frac{G_{11}(k_1,s)}{G_{11}(\sigma(s),s)}.$$

Therefore,

$$G(t,s) \ge \frac{G_{11}(k_1,s)}{G_{11}(\sigma(s),s)} G(\sigma(s),s).$$

Case (ii). For $t_1 \le t \le t_2 < s < t_3 < \sigma^3(t_4)$

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{13}(t,s)}{G_{13}(\sigma(s),s)} = \frac{[(t_1t_2^2 - t_1^2t_2) + t(t_1^2 - t_2^2) + t^2(t_2 - t_1)]}{[(t_1t_2^2 - t_1^2t_2) + \sigma(s)(t_1^2 - t_2^2) + (\sigma(s))^2(t_2 - t_1)]}.$$

From $(A2)$, we have $G_{13}(t,s) \le G_{13}(\sigma(s),s)$. Therefore, $G(t,s) \le G(\sigma(s),s)$. And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{13}(t,s)}{G_{13}(\sigma(s),s)} \ge \frac{G_{13}(\frac{t_1}{2},s)}{G_{13}(\sigma(s),s)}.$$

Therefore,

$$G(t,s) \ge \frac{G_{13}(\frac{t_1}{2},s)}{G_{13}(\sigma(s),s)} G(\sigma(s),s).$$

Case (iii). For $t_1 \le t \le t_2 < t_3 < s < \sigma^3(t_4)$

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{14}(t,s)}{G_{14}(\sigma(s),s)} = \frac{[-(t_1t_2^2 - t_1^2t_2) - t(t_1^2 - t_2^2) - t^2(t_2 - t_1)]}{[-(t_1t_2^2 - t_1^2t_2) - \sigma(s)(t_1^2 - t_2^2) - (\sigma(s))^2(t_2 - t_1)]}.$$

From $(A2)$, we have $G_{14}(t,s) \le G_{14}(\sigma(s),s)$. Therefore, $G(t,s) \le G(\sigma(s),s)$. And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{G_{14}(t,s)}{G_{14}(\sigma(s),s)} \ge \frac{G_{14}(\frac{t_1}{2},s)}{G_{14}(\sigma(s),s)}.$$

Therefore,

$$G(t,s) \ge \frac{G_{14}(\frac{t_1}{2},s)}{G_{14}(\sigma(s),s)} G(\sigma(s),s).$$

Case (iv). For $t_1 \le t < s \le t_2 < t_3 < \sigma^3(t_4)$
from $(A2)$ and case (ii) we have, $G_{12}(t,s) \le G_{12}(\sigma(s),s)$. Therefore, $G(t,s) \le G(\sigma(s),s)$. And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} \ge \min\left\{\frac{G_{12}(k_2,s)}{G_{12}(\sigma(s),s)}, \frac{G_{13}(\frac{t_1}{2},s)}{G_{13}(\sigma(s),s)}\right\}.$$

Therefore,

$$G(t,s) \ge \min\left\{\frac{G_{12}(k_2,s)}{G_{12}(\sigma(s),s)}, \frac{G_{13}(\frac{t_1}{2},s)}{G_{13}(\sigma(s),s)}\right\} G(\sigma(s),s).$$

Case (v). For $t_1 < t_2 \le \sigma(s) < t < t_3 < \sigma^3(t_4)$
from $(A2)$ and case (i), we have $G_{22}(t,s) \le G_{22}(\sigma(s),s)$. Therefore, $G(t,s) \le G(\sigma(s),s)$. And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} \ge \min\left\{\frac{G_{11}(k_1,s)}{G_{11}(\sigma(s),s)}, \frac{G_{22}(k_2,s)}{G_{22}(\sigma(s),s)}\right\}.$$

Therefore,

$$G(t,s) \ge \min\left\{\frac{G_{11}(k_1,s)}{G_{11}(\sigma(s),s)}, \frac{G_{22}(k_2,s)}{G_{22}(\sigma(s),s)}\right\} G(\sigma(s),s).$$

Case (vi). For $t_1 < t_2 < t_3 \leq \sigma(s) < t \leq \sigma^3(t_4)$
from $(A3)$ and case (v), we have $G_{33}(t,s) \leq G_{33}(\sigma(s),s)$. Therefore, $G(t,s) \leq G(\sigma(s),s)$. And also, from $(A2)$, we have

$$\frac{G(t,s)}{G(\sigma(s),s)} \geq \min\left\{\frac{G_{11}(k_1,s)}{G_{11}(\sigma(s),s)}, \frac{G_{22}(k_2,s)}{G_{22}(\sigma(s),s)}, \frac{G_{33}(\frac{t_1}{2},s)}{G_{33}(\sigma(s),s)}\right\}.$$

Therefore,

$$G(t,s) \geq \min\left\{\frac{G_{11}(k_1,s)}{G_{11}(\sigma(s),s)}, \frac{G_{22}(k_2,s)}{G_{22}(\sigma(s),s)}, \frac{G_{33}(\frac{t_1}{2},s)}{G_{33}(\sigma(s),s)}\right\} G(\sigma(s),s).$$

In other Cases, the inequality can be established similarly and so their arguments omitted. By consolidating all the above cases, we get

$$mG(\sigma(s),s) \leq G(t,s) \leq G(\sigma(s),s), \text{ for all } (t,s) \in [t_1, \sigma^3(t_4)]_{\mathbb{T}} \times [t_1, t_4],$$

where m given in (2.5). □

Let $y(t)$ be the solution of the BVP (1.1)-(1.2), and is given by

$$y(t) = \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s, \text{ for all } t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}. \tag{2.7}$$

Define

$$X = \left\{ \ u : u \in C^3_{rd}[t_1, \sigma^3(t_4)]_{\mathbb{T}} \ \right\}.$$

The set of functions $u : \mathbb{T} \to \mathbb{R}$ that are three times differentiable and whose derivative is rd-continuous is denoted by C^3_{rd}, with norm,

$$\| u \| = \max_{t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}} | u(t) | .$$

Then $(X, \| . \|)$ is a Banach space. Define a set κ by

$$\kappa = \left\{ u \in X : u(t) \geq 0 \text{ on } [t_1, \sigma^3(t_4)]_{\mathbb{T}} \text{ and } \min_{t \in \ [t_1, \sigma^3(t_4)]_{\mathbb{T}}} u(t) \geq m \| u \| \right\}. \tag{2.8}$$

then it is easy to see that κ is a positive cone in X. Now we define the operator $T : \kappa \to X$ by

$$Ty(t) = \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s, \ t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}. \tag{2.9}$$

If $y \in \kappa$ is a fixed point of T, then y satisfies (2.7) and hence y is a positive solution of the BVP (1.1)-(1.2). We seek the fixed points of the operator T in the cone κ.

Lemma 2.1. *The operator T defined in* (2.9) *is self map on κ.*

Proof. Let $y \in \kappa$. From (2.4) we have $Ty(t) \geq 0$ for all $t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}$.

$$\begin{aligned} Ty(t) &= \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s \geq \lambda \int_{t_1}^{\sigma(t_4)} mG(\sigma(s),s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s \\ &\geq \lambda m \int_{t_1}^{\sigma(t_4)} \max_{t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s \\ &\geq m \max_{t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}} \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s = m \| Ty \| . \end{aligned}$$

Therefore,

$$\min_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} Ty(t) \geq m \parallel Ty \parallel .$$

Also, from the positivity of $G(t,s)$, it clear that for $y \in \kappa$, that $Ty(t) \geq 0,\ t_1 \leq t \leq \sigma^3(t_4)$, and so $Ty \in \kappa$; thus $T : \kappa \to \kappa$. Further arguments yield that T is completely continuous. □

3. Existence of positive solutions

In this section we determine the eigenvalue intervals for which the four point BVP (1.1)-(1.2) possess a positive solution by using Krasnosel'skii fixed point theorem on cone.

Theorem 3.2. [Krasnosel'skii] [20] *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous operator such that either*
(i) $\parallel Tu \parallel \leq \parallel u \parallel,\ u \in K \cap \partial\Omega_1$ and $\parallel Tu \parallel \geq \parallel u \parallel,\ u \in K \cap \partial\Omega_2$, or
(ii) $\parallel Tu \parallel \geq \parallel u \parallel,\ u \in K \cap \partial\Omega_1$ and $\parallel Tu \parallel \leq \parallel u \parallel,\ u \in K \cap \partial\Omega_2$
holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.3. *Assume that conditions $(A1) - (A4)$ are satisfied and if*

$$\frac{1}{m^2[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s] f_\infty} < \lambda < \frac{1}{[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s] f^0}. \tag{3.10}$$

Then the BVP (1.1)-(1.2) *has at least one positive solution lies in κ.*

Proof. Let λ be given as in (3.10) and let $\epsilon > 0$ be such that

$$\frac{1}{m^2[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s](f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s](f^0 + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined in (2.9). By the definition of f^0, there exist $H_i^1 > 0,\ i = 0,1,2$ such that

$$\sup_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t,y,y^\Delta,y^{\Delta^2})}{y} \leq (f^0 + \epsilon),$$

for $0 < y \leq H_0^1,\ 0 < y^\Delta \leq H_1^1, 0 < y^{\Delta^2} \leq H_2^1$. Let $H^1 = \min\left\{H_i^1 : i = 0,1,2\right\}$. It follows that

$$f(t,y,y^\Delta,y^{\Delta^2}) \leq (f^0 + \epsilon)y, \text{ for } 0 < y, y^\Delta, y^{\Delta^2} \leq H^1.$$

Let us choose $y \in \kappa$ with $\parallel y \parallel = H^1$. Then, we have from (2.9),

$$\begin{aligned} Ty(t) &= \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s,y,y^\Delta,y^{\Delta^2})\Delta s \leq \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s) f(s,y,y^\Delta,y^{\Delta^2})\Delta s \\ &\leq \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)(f^0+\epsilon) y(s)\Delta s \leq \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)(f^0+\epsilon) \parallel y \parallel \Delta s \\ &\leq \parallel y \parallel,\ t \in [t_1, \sigma^3(t_4)]. \end{aligned}$$

Therefore, $\parallel Ty \parallel \leq \parallel y \parallel$. Hence, if we set

$$\Omega_1 = \{u \in X : \parallel u \parallel < H^1\},$$

then

$$\parallel Ty \parallel \leq \parallel y \parallel, \text{ for } y \in \kappa \cap \partial\Omega_1. \tag{3.11}$$

By the definition of f_∞, there exist $\overline{H}_i^2 > 0,\ i = 0,1,2$ such that

$$\inf_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t,y,y^\Delta,y^{\Delta^2})}{y} \geq (f_\infty - \epsilon),\ \text{for } y \geq \overline{H}_0^2, y^\Delta \geq \overline{H}_1^2, y^{\Delta^2} \geq \overline{H}_2^2.$$

Let $\overline{H}^2 = \max\left\{\overline{H}_i^2 : i = 0,1,2\right\}$, it follows that $f(t,y,y^\Delta,y^{\Delta^2}) \geq (f_\infty - \epsilon)y$, for $y, y^\Delta, y^{\Delta^2} \geq \overline{H}^2$. If we set $H^2 = \max\left\{2H^1, \frac{1}{m}\overline{H}^2\right\}$, and define $\Omega_2 = \{u \in X :\| u \|< H^2\}$. If $y \in \kappa \cap \partial\Omega_2,$ so that $\| y \|= H^2$, then

$$\min_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} y(t) \geq m \| y \| \geq \overline{H}^2.$$

And we have

$$\begin{aligned} Ty(t) &= \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s,y,y^\Delta,y^{\Delta^2})\Delta s \geq m\lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s) f(s,y,y^\Delta,y^{\Delta^2})\Delta s \\ &\geq m\lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)(f_\infty - \epsilon) y(s)\Delta s \geq m^2\lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)(f_\infty - \epsilon) \| y \| \Delta s \\ &\geq \| y \| . \end{aligned}$$

Thus, $\| Ty \| \geq \| y \|$, and so

$$\| Ty \| \geq \| y \|,\ \text{for } y \in \kappa \cap \partial\Omega_2. \tag{3.12}$$

An application of Theorem 3.2 to (3.11) and (3.12) yields a fixed point of T that lies in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the positive solution of the BVP (1.1)-(1.2). □

Theorem 3.4. *Assume that conditions* $(A1) - (A4)$ *are satisfied and if*

$$\frac{1}{m^2[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s] f_0} < \lambda < \frac{1}{[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s] f^\infty}. \tag{3.13}$$

Then the BVP (1.1)-(1.2) *has at least one solution lies in* κ.

Proof. Let λ be given as in (3.13) and let $\epsilon > 0$ be such that

$$\frac{1}{m^2[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s](f_0 - \epsilon)} \leq \lambda \leq \frac{1}{[\int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)\Delta s](f^\infty + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined in (2.9). By the definition of f_0, there exist $J_i^1 > 0,\ i = 0,1,2$ such that

$$\inf_{t\in[t_1,\sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t,y,y^\Delta,y^{\Delta^2})}{y} \geq (f_0 - \epsilon),\ \text{for } 0 < y \leq J_0^1, 0 < y^\Delta \leq J_1^1, 0 < y^{\Delta^2} \leq J_2^1.$$

Let $J^1 = \min\left\{J_i^1 : i = 0,1,2\right\}$. It follows that

$$f(t,y,y^\Delta,y^{\Delta^2}) \geq (f_0 - \epsilon)y,\ \text{for } 0 < y, y^\Delta, y^{\Delta^2} \leq J^1.$$

In this case, define $\Omega_1 = \{u \in X :\| u \|< J^1\}$.

Then, for $y \in \kappa \cap \partial\Omega_1$, we have $f(s, y, y^{\Delta}, y^{\Delta^2}) \geq (f_0 - \epsilon)y(s),\ s \in [t_1, t_4]$, and moreover, $y(t) \geq m \parallel y \parallel$, $t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}$. Thus

$$Ty(t) = \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s \geq \lambda \int_{t_1}^{\sigma(t_4)} mG(\sigma(s), s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s$$
$$\geq m\lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s), s)(f_0 - \epsilon) y(s) \Delta s \geq m^2 \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s), s)(f_0 - \epsilon) \parallel y \parallel \Delta s \geq \parallel y \parallel .$$

Hence,

$$\parallel Ty \parallel \geq \parallel y \parallel, \text{ for } y \in \kappa \cap \partial\Omega_1. \tag{3.14}$$

It remains for us to consider f^{∞}. By the definition of f^{∞}, there exist $\overline{J}_i^2 > 0,\ i = 0, 1, 2$ such that

$$\sup_{t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}} \frac{f(t, y, y^{\Delta}, y^{\Delta^2})}{y} \leq (f^{\infty} + \epsilon), \text{ for } y \geq \overline{J}_0^2, y^{\Delta} \geq \overline{J}_1^2, y^{\Delta^2} \geq \overline{J}_2^2.$$

Let $\overline{J}^2 = \max \left\{ \overline{J}_i^2 : i = 0, 1, 2 \right\}$. It follows that
$f(t, y, y^{\Delta}, y^{\Delta^2}) \leq (f^{\infty} + \epsilon)y$, for $y, y^{\Delta}, y^{\Delta^2} \geq \overline{J}^2$. There are two subcases.
Case (i). f is bounded. Suppose $L > 0$ is such that

$$\max_{t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}} f(t, y, y^{\Delta}, y^{\Delta^2}) \leq L,$$

for all $0 < y, y^{\Delta}, y^{\Delta^2} < \infty$.

$$\text{Let } J^2 = \max \left\{ 2J^1, L\lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s), s) \Delta s \right\}$$

and let

$$\Omega_2 = \{ u \in X : \parallel u \parallel < J^2 \}.$$

Then, for $y \in \kappa \cap \partial\Omega_2$, we have

$$Ty(t) = \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s \leq \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s), s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s$$
$$\leq L\lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s), s) \Delta s \leq \parallel y \parallel,\ t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}},$$

and so

$$\parallel Ty \parallel \leq \parallel y \parallel, \text{ for } y \in \kappa \cap \partial\Omega_2. \tag{3.15}$$

Case (ii). f is unbounded. Let $J_i^2 > \max\{2J_i^1, \overline{J}_i^2\},\ i = 0, 1, 2$ be such that $f(t, y, y^{\Delta}, y^{\Delta^2}) \leq f(t, J_0^2, J_1^2, J_2^2)$, for $0 < y \leq J_0^2, 0 < y^{\Delta} \leq J_1^2, 0 < y^{\Delta^2} \leq J_2^2$. Let $J^2 = \max\{J_i^2 : i = 0, 1, 2\}$, and let

$$\Omega_2 = \{ u \in X : \parallel u \parallel < J^2 \}.$$

Choosing $y \in \kappa \cap \partial\Omega_2$,

$$\begin{aligned} Ty(t) &= \lambda \int_{t_1}^{\sigma(t_4)} G(t,s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s \leq \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s) f(s, y, y^{\Delta}, y^{\Delta^2}) \Delta s \\ &\leq \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s) f(s, J_0^2, J_1^2, J_2^2) \Delta s \leq \lambda \int_{t_1}^{\sigma(t_4)} G(\sigma(s),s)(f^{\infty} + \epsilon) \parallel J^2 \parallel \Delta s \\ &\leq J^2 = \parallel y \parallel, \ t \in [t_1, \sigma^3(t_4)]_{\mathbb{T}}. \end{aligned}$$

And so

$$\parallel Ty \parallel \leq \parallel y \parallel, \ \text{for } y \in \kappa \cap \partial\Omega_2. \tag{3.16}$$

An application of Theorem 3.2, to (3.14), (3.15) and (3.16) yields a fixed point of T that lies in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the positive solution of the BVP (1.1)-(1.2). □

We demonstrate our results with the following examples.

Example 3.1. We consider the following boundary value problem

$$y^{\Delta^3} + \lambda y(25 - 24.5e^{-13y})(30 - 29.5e^{-5y^{\Delta}})(71 - 70e^{-3y^{\Delta^2}}) = 0, t \in [0,1] \cap \mathbb{T} \tag{3.17}$$

where $\mathbb{T} = \left\{0, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}\right\} \cup [1,2]$, with the boundary conditions

$$y(0) = 0, \ y\left(\frac{4}{9}\right) = 0, \ \frac{9}{5} y\left(\frac{2}{3}\right) - \frac{2}{3} y\left(\sigma^3(1)\right) = 0. \tag{3.18}$$

The Green's function for the homogeneous BVP is given by

$$\underset{t \in [0, \frac{4}{9}]}{G(t,s)} = \begin{cases} G_{11}(t,s), & 0 \leq \sigma(\mathrm{s}) < \mathrm{t} \leq \frac{4}{9} < \frac{2}{3} < \sigma^3(1) \\ G_{12}(t,s), & 0 \leq \mathrm{t} < \mathrm{s} \leq \frac{4}{9} < \frac{2}{3} < \sigma^3(1) \\ G_{13}(t,s), & 0 \leq \mathrm{t} \leq \frac{4}{9} < \mathrm{s} < \frac{2}{3} < \sigma^3(1) \\ G_{14}(t,s), & 0 \leq \mathrm{t} \leq \frac{4}{9} < \frac{2}{3} < \mathrm{s} < \sigma^3(1) \end{cases}$$

$$\underset{t \in [\frac{4}{9}, \frac{2}{3}]}{G(t,s)} = \begin{cases} G_{21}(t,s), & 0 < \sigma(\mathrm{s}) < \frac{4}{9} \leq \mathrm{t} \leq \frac{2}{3} < \sigma^3(1) \\ G_{22}(t,s), & 0 < \frac{4}{9} \leq \sigma(\mathrm{s}) < \mathrm{t} < \frac{2}{3} < \sigma^3(1) \\ G_{23}(t,s), & 0 < \frac{4}{9} \leq \mathrm{t} < \mathrm{s} \leq \frac{2}{3} < \sigma^3(1) \\ G_{24}(t,s), & 0 < \frac{4}{9} \leq \mathrm{t} \leq \frac{2}{3} < \mathrm{s} < \sigma^3(1) \end{cases}$$

$$\underset{t \in [\frac{2}{3}, \sigma^3(1)]}{G(t,s)} = \begin{cases} G_{31}(t,s), & 0 < \sigma(\mathrm{s}) < \frac{4}{9} < \frac{2}{3} \leq \mathrm{t} \leq \sigma^3(1) \\ G_{32}(t,s), & 0 < \frac{4}{9} < \sigma(\mathrm{s}) < \frac{2}{3} \leq \mathrm{t} \leq \sigma^3(1) \\ G_{33}(t,s), & 0 < \frac{4}{9} < \frac{2}{3} \leq \sigma(\mathrm{s}) < \mathrm{t} \leq \sigma^3(1) \\ G_{34}(t,s), & 0 < \frac{4}{9} < \frac{2}{3} \leq \mathrm{t} < \mathrm{s} \leq \sigma^3(1) \end{cases}$$

where

$$G_{11}(t,s)=G_{21}(t,s)=\left[\frac{-1}{2}-\frac{7425}{104}t+\frac{567}{4}t^2\right]\left[(\sigma(s))^2\right]=G_{31}(t,s)$$

$$G_{12}(t,s)=\left[\frac{729}{8}t-\frac{243}{2}t^2\right]\left[(\frac{4}{9}-\sigma(s))^2\right]+\left[-\frac{2160}{11}t+\frac{1215}{4}t^2\right]$$
$$\left[\frac{17}{15}\sigma(s)\sigma^2(s)-\frac{8}{27}(\sigma(s)+\sigma^2(s))+\frac{2}{15}\right]$$

$$G_{13}(t,s)=G_{23}(t,s)=\left[-\frac{2160}{11}t+\frac{1215}{4}t^2\right]\left[\frac{17}{15}\sigma(s)\sigma^2(s)-\frac{8}{27}(\sigma(s)+\sigma^2(s))+\frac{2}{15}\right]$$

$$G_{14}(t,s)=G_{24}(t,s)=\left[\frac{1440}{11}t-\frac{405}{2}t^2\right]\left[(\sigma^3(1)-\sigma(s))^2\right]=G_{34}(t,s)$$

$$G_{22}(t,s)=G_{32}(t,s)=\left[\frac{-1}{2}-\frac{7425}{104}t+\frac{567}{4}t^2][(\sigma(s))^2\right]+\left[\frac{729}{8}t-\frac{243}{2}t^2\right]\left[(\frac{4}{9}-\sigma(s))^2\right]$$

$$G_{33}(t,s)=\left[\frac{-1}{2}-\frac{7425}{104}t+\frac{567}{4}t^2\right]\left[(\sigma(s))^2\right]+\left[\frac{729}{8}t-\frac{243}{2}t^2\right]\left[(\frac{4}{9}-\sigma(s))^2\right]$$
$$+\left[\frac{3888}{11}t-\frac{2187}{4}t^2\right]\left[(\frac{2}{3}-\sigma(s))^2\right].$$

We found that $m = 0.06$, $f_\infty = 53250$ and $f^0 = 0.25$. Employing Theorem 3.3, we get the eigenvalue interval $0.020254 < \lambda < 0.38751$, for which (3.17)-(3.18) has at least one positive solution.

Example 3.2. We consider the following boundary value problem

$$y^{\Delta^3}+\lambda y(20-19.5e^{-7y})(25-24e^{-5y^\Delta})(72-71e^{-3y^{\Delta^2}})=0, t\in[0,1]\cap\mathbb{T} \tag{3.19}$$

where $\mathbb{T}=\left\{0,\frac{1}{9},\frac{2}{9},\frac{1}{3},\frac{4}{9},\frac{5}{9},\frac{2}{3},\frac{7}{9},\frac{8}{9}\right\}\cup[1,2]$, with the boundary conditions

$$y(0)=0,\ y\left(\frac{4}{9}\right)=0,\ \frac{9}{5}y\left(\frac{2}{3}\right)-\frac{2}{3}y\left(\sigma^3(1)\right)=0. \tag{3.20}$$

After computation, we found that $m = 0.06$, $f_0 = 0.5$ and $f^\infty = 36000$. Employing Theorem 3.4, we get the eigenvalue interval $0.0000454 < \lambda < 0.03739$, for which (3.19), (3.20) has at least one positive solution.

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