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Some new sequences that converge to a generalization of Euler's constant

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ABSTRACT. The purpose of this paper is to study the properties of some sequences that converge quickly to a generalization of Euler's constant, i.e. the limit of the sequence

$$\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}},$$

where $a\in(0,+\infty)$.

1. INTRODUCTION

Euler's constant, being one of the most important constants in mathematics, was investigated by many mathematicians. Usually denoted by γ, this constant is the limit of the sequence $(D_n)_{n\in\mathbb{N}}$ defined by $D_n=1+1/2+\cdots+1/n-\ln n$, for each $n\in\mathbb{N}$. It is well-known that $\lim_{n\to\infty} n(D_n-\gamma)=1/2$ (see [1], [3], [4], [6, pp. 73–75], [8], [13, Problem 18, pp. 38, 197], [14], [22], [24], [25], [26], [27]).

In order to increase the slow rate of convergence of the sequence $(D_n)_{n\in\mathbb{N}}$ to γ, D. W. DeTemple considered in [5] the sequence $(R_n)_{n\in\mathbb{N}}$ defined by

$$R_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\left(n+\frac{1}{2}\right),$$

for each $n\in\mathbb{N}$, and he proved that $\frac{1}{24(n+1)^2}<R_n-\gamma<\frac{1}{24n^2}$, for each $n\in\mathbb{N}$.

L. Tóth used in [23] the sequence $(T_n)_{n\in\mathbb{N}}$ defined by

$$T_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\left(n+\frac{1}{2}+\frac{1}{24n}\right),$$

for each $n\in\mathbb{N}$, and T. Negoi proved in [12] that $\frac{1}{48(n+1)^3}<\gamma-T_n<\frac{1}{48n^3}$, for each $n\in\mathbb{N}$.

Let $a\in(0,+\infty)$ and let $(y_n(a))_{n\in\mathbb{N}}$ be the sequence defined by

$$y_n(a)=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a},$$

for each $n\in\mathbb{N}$. The sequence $(y_n(a))_{n\in\mathbb{N}}$ converges (see, for example, [7, p. 453]; see also [15], [16], [17], [18], [19], [20] and some of the references therein) and its limit, denoted by

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$\gamma(a)$, can be viewed as a generalization of Euler's constant. Clearly, $\gamma(1) = \gamma$. One can prove that $\lim_{n\to\infty} n(y_n(a) - \gamma(a)) = \frac{1}{2}$,

$$\lim_{n\to\infty} n^2 \left(\sum_{k=1}^{n} \frac{1}{a+k-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a} \right) - \gamma(a) \right) = \frac{1}{24}$$

and

$$\lim_{n\to\infty} n^3 \left(\gamma(a) - \sum_{k=1}^{n} \frac{1}{a+k-1} + \ln\left(\frac{a+n-1}{a} + \frac{1}{2a} + \frac{1}{24a(a+n-1)} \right) \right) = \frac{1}{48}.$$

See [15], [16], [17], [18], [19], [20] and [21] for the above-mentioned results, as well as for other results regarding the generalization of Euler's constant $\gamma(a)$.

We mention that, for recent papers on Euler's constant, the interested reader is referred to [2], [9], [10] and [11].

In Section 2 we obtain a significant improvement of the convergence to $\gamma(a)$, considering a new sequence $(\alpha_n(a))_{n\geq 2}$, with the argument of the logarithmic term modified. We prove that this sequence converges to $\gamma(a)$ with the speed of convergence n^{-4}. We obtain, based on the sequence $(\alpha_n(a))_{n\geq 2}$, other sequences that converge quickly to $\gamma(a)$.

2. Sequences that converge to $\gamma(a)$

Theorem 2.1. *Let $a \in (0, +\infty)$. We specify that $\gamma(a)$ is the limit of the sequence $(y_n(a))_{n\in\mathbb{N}}$ from Introduction.*

(i) *We consider the sequence $(\alpha_n(a))_{n\geq 2}$ defined by*

$$\alpha_n(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} - \ln\left(\frac{a+n-1}{a} - \frac{1}{12a(a+n-1)} \right),$$

for each $n \in \mathbb{N} \setminus \{1\}$. Then

$$\lim_{n\to\infty} n^4(\alpha_n(a) - \gamma(a)) = \frac{17}{1440}.$$

(ii) *We consider the sequence $(\beta_n(a))_{n\geq 2}$ defined by*

$$\beta_n(a) = \alpha_n(a) - \frac{17}{1440(a+n-1)^4},$$

for each $n \in \mathbb{N} \setminus \{1\}$. Then

$$\lim_{n\to\infty} n^6(\gamma(a) - \beta_n(a)) = \frac{137}{36288}.$$

(iii) *We consider the sequence $(\delta_n(a))_{n\geq 2}$ defined by*

$$\delta_n(a) = \beta_n(a) + \frac{137}{36288(a+n-1)^6},$$

for each $n \in \mathbb{N} \setminus \{1\}$. Then

$$\lim_{n\to\infty} n^8(\delta_n(a) - \gamma(a)) = \frac{1733}{414720}.$$

Proof. (i) We have

$$\begin{aligned}
&\alpha_n(a)-\alpha_{n+1}(a)\\
&=-\frac{1}{2(a+n-1)}-\frac{1}{2(a+n)}\\
&\quad-\ln\left(a+n-1-\frac{1}{12(a+n-1)}\right)+\ln\left(a+n-\frac{1}{12(a+n)}\right)\\
&=-\frac{1}{2(a+n)\left(1-\frac{1}{a+n}\right)}-\frac{1}{2(a+n)}\\
&\quad-\ln\left(1-\frac{1}{a+n}-\frac{1}{12(a+n)^2\left(1-\frac{1}{a+n}\right)}\right)+\ln\left(1-\frac{1}{12(a+n)^2}\right),
\end{aligned}$$

for each $n\in\mathbb{N}\setminus\{1\}$. Set $\varepsilon_n:=\dfrac{1}{a+n}$, for each $n\in\mathbb{N}\setminus\{1\}$. Since $\varepsilon_n\in(-1,1)$, $-\varepsilon_n-\dfrac{1}{12}\cdot\dfrac{\varepsilon_n^2}{1-\varepsilon_n}\in(-1,1]$ and $-\dfrac{1}{12}\varepsilon_n^2\in(-1,1]$, for each $n\in\mathbb{N}\setminus\{1\}$, using the series expansion ([7, pp. 171–179, p. 209]) we obtain

$$\begin{aligned}
&\alpha_n(a)-\alpha_{n+1}(a)\\
&=-\frac{1}{2}\cdot\frac{\varepsilon_n}{1-\varepsilon_n}-\frac{1}{2}\,\varepsilon_n-\ln\left(1-\varepsilon_n-\frac{1}{12}\cdot\frac{\varepsilon_n^2}{1-\varepsilon_n}\right)+\ln\left(1-\frac{1}{12}\,\varepsilon_n^2\right)\\
&=-\frac{1}{2}\sum_{k=1}^{\infty}\varepsilon_n^k-\frac{1}{2}\,\varepsilon_n+\sum_{k=1}^{\infty}\frac{\varepsilon_n^k}{k}\left(1+\frac{1}{12}\cdot\frac{\varepsilon_n}{1-\varepsilon_n}\right)^k-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{1}{12}\,\varepsilon_n^2\right)^k\\
&=\frac{17}{360}\,\varepsilon_n^5+\frac{17}{144}\,\varepsilon_n^6+\frac{1291}{6048}\,\varepsilon_n^7+\frac{577}{1728}\,\varepsilon_n^8+\frac{5009}{10368}\,\varepsilon_n^9+\frac{853}{1280}\,\varepsilon_n^{10}+O(\varepsilon_n^{11}),
\end{aligned}$$

for each $n\in\mathbb{N}\setminus\{1\}$.

Now, according to the Stolz-Cesaro Theorem, the $0/0$ case, we get

$$\begin{aligned}
\lim_{n\to\infty}n^4(\alpha_n(a)-\gamma(a))&=\lim_{n\to\infty}\frac{\alpha_n(a)-\gamma(a)}{\frac{1}{n^4}}\\
&=\lim_{n\to\infty}\frac{\alpha_{n+1}(a)-\gamma(a)-(\alpha_n(a)-\gamma(a))}{\frac{1}{(n+1)^4}-\frac{1}{n^4}}\\
&=\lim_{n\to\infty}\frac{\alpha_n(a)-\alpha_{n+1}(a)}{\frac{1}{n^4}-\frac{1}{(n+1)^4}}=\lim_{n\to\infty}\frac{\frac{17}{360}\,\varepsilon_n^5+O(\varepsilon_n^6)}{\frac{4}{n^5}+O\left(\frac{1}{n^6}\right)}=\frac{17}{1440}.
\end{aligned}$$

(ii) We are able to write that

$$\begin{aligned}
&\beta_{n+1}(a)-\beta_n(a)\\
&=\alpha_{n+1}(a)-\alpha_n(a)-\frac{17}{1440(a+n)^4}+\frac{17}{1440(a+n-1)^4}\\
&=\alpha_{n+1}(a)-\alpha_n(a)-\frac{17}{1440}\,\varepsilon_n^4+\frac{17}{1440}\cdot\frac{\varepsilon_n^4}{(1-\varepsilon_n)^4}\\
&=\frac{137}{6048}\,\varepsilon_n^7+\frac{137}{1728}\,\varepsilon_n^8+\frac{9227}{51840}\,\varepsilon_n^9+\frac{1249}{3840}\,\varepsilon_n^{10}+O(\varepsilon_n^{11}),
\end{aligned}$$

for each $n\in\mathbb{N}\setminus\{1\}$.

Now, according to the Stolz-Cesaro Theorem, the $0/0$ case, we get

$$\lim_{n\to\infty} n^6(\gamma(a)-\beta_n(a)) = \lim_{n\to\infty}\frac{\gamma(a)-\beta_n(a)}{\frac{1}{n^6}}$$
$$= \lim_{n\to\infty}\frac{\gamma(a)-\beta_{n+1}(a)-(\gamma(a)-\beta_n(a))}{\frac{1}{(n+1)^6}-\frac{1}{n^6}}$$
$$= \lim_{n\to\infty}\frac{\beta_{n+1}(a)-\beta_n(a)}{\frac{1}{n^6}-\frac{1}{(n+1)^6}} = \lim_{n\to\infty}\frac{\frac{137}{6048}\,\varepsilon_n^7+O(\varepsilon_n^8)}{\frac{6}{n^7}+O\left(\frac{1}{n^8}\right)} = \frac{137}{36288}.$$

(iii) We are able to write that

$$\delta_n(a)-\delta_{n+1}(a)$$
$$= \beta_n(a)-\beta_{n+1}(a)+\frac{137}{36288(a+n-1)^6}-\frac{137}{36288(a+n)^6}$$
$$= \beta_n(a)-\beta_{n+1}(a)+\frac{137}{36288}\cdot\frac{\varepsilon_n^6}{(1-\varepsilon_n)^6}-\frac{137}{36288}\,\varepsilon_n^6$$
$$= \frac{1733}{51840}\,\varepsilon_n^9+\frac{1733}{11520}\,\varepsilon_n^{10}+O(\varepsilon_n^{11}),$$

for each $n\in\mathbb{N}\setminus\{1\}$.

Now, according to the Stolz-Cesaro Theorem, the $0/0$ case, we get

$$\lim_{n\to\infty} n^8(\delta_n(a)-\gamma(a)) = \lim_{n\to\infty}\frac{\delta_n(a)-\gamma(a)}{\frac{1}{n^8}}$$
$$= \lim_{n\to\infty}\frac{\delta_{n+1}(a)-\gamma(a)-(\delta_n(a)-\gamma(a))}{\frac{1}{(n+1)^8}-\frac{1}{n^8}}$$
$$= \lim_{n\to\infty}\frac{\delta_n(a)-\delta_{n+1}(a)}{\frac{1}{n^8}-\frac{1}{(n+1)^8}} = \lim_{n\to\infty}\frac{\frac{1733}{51840}\,\varepsilon_n^9+O(\varepsilon_n^{10})}{\frac{8}{n^9}+O\left(\frac{1}{n^{10}}\right)} = \frac{1733}{414720}.$$

□

In the same manner as in the proof of Theorem 2.1, considering the sequence in each of the following parts, we get the indicated limit:

$\eta_n(a)=\delta_n(a)-\dfrac{1733}{414720(a+n-1)^8}$, for each $n\in\mathbb{N}\setminus\{1\}$,

$$\lim_{n\to\infty} n^{10}(\gamma(a)-\eta_n(a)) = \frac{103669}{13685760};$$

$\theta_n(a)=\eta_n(a)+\dfrac{103669}{13685760(a+n-1)^{10}}$, for each $n\in\mathbb{N}\setminus\{1\}$,

$$\lim_{n\to\infty} n^{12}(\theta_n(a)-\gamma(a)) = \frac{171943367}{8151736320};$$

$\lambda_n(a)=\theta_n(a)-\dfrac{171943367}{8151736320(a+n-1)^{12}}$, for each $n\in\mathbb{N}\setminus\{1\}$,

$$\lim_{n\to\infty} n^{14}(\gamma(a)-\lambda_n(a)) = \frac{20901887}{250822656};$$

$$\mu_n(a) = \lambda_n(a) + \frac{20901887}{250822656(a+n-1)^{14}}, \text{ for each } n \in \mathbb{N} \setminus \{1\},$$

$$\lim_{n\to\infty} n^{16}(\mu_n(a) - \gamma(a)) = \frac{129603649621}{292387553280};$$

$$\nu_n(a) = \mu_n(a) - \frac{129603649621}{292387553280(a+n-1)^{16}}, \text{ for each } n \in \mathbb{N} \setminus \{1\},$$

$$\lim_{n\to\infty} n^{18}(\gamma(a) - \nu_n(a)) = \frac{18862007058299}{6176257081344};$$

$$\sigma_n(a) = \nu_n(a) + \frac{18862007058299}{6176257081344(a+n-1)^{18}}, \text{ for each } n \in \mathbb{N} \setminus \{1\},$$

$$\lim_{n\to\infty} n^{20}(\sigma_n(a) - \gamma(a)) = \frac{900954407043127}{34054550323200}.$$

We remark the pattern in forming the sequences from Theorem 2.1 and those mentioned above. For example, the general term of the sequence $(\sigma_n(a))_{n\geq 2}$ can be written in the form

$$\begin{aligned}\sigma_n(a) = {} & \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \frac{1}{2(a+n-1)} \\ & - \ln\left(\frac{a+n-1}{a} - \frac{B_2}{2}\cdot\frac{1}{a(a+n-1)}\right) \\ & + \sum_{k=2}^{9}\left(\frac{B_{2k}}{2k} - \frac{1}{k}\left(\frac{B_2}{2}\right)^k\right)\frac{1}{(a+n-1)^{2k}},\end{aligned}$$

where B_{2k} is the $2k$th Bernoulli number. Related to this remark, see also [16, Remark 3.4], [19, p. 71, Remark 2.1.3; pp. 100, 101, Remark 3.1.6].

For Euler's constant $\gamma = 0.5772156649\ldots$ we obtain, for example:

$\alpha_2(1) = 0.5779062286\ldots$; $\alpha_3(1) = 0.5773567706\ldots$;
$\beta_2(1) = 0.5771683814\ldots$; $\beta_3(1) = 0.5772110230\ldots$;
$\delta_2(1) = 0.5772273713\ldots$; $\delta_3(1) = 0.5772162018\ldots$;
$\eta_2(1) = 0.5772110481\ldots$; $\eta_3(1) = 0.5772155649\ldots$;
$\theta_2(1) = 0.5772184455\ldots$; $\theta_3(1) = 0.5772156932\ldots$;
$\lambda_2(1) = 0.5772132959\ldots$; $\lambda_3(1) = 0.5772156535\ldots$;
$\mu_2(1) = 0.5772183822\ldots$; $\mu_3(1) = 0.5772156709\ldots$;
$\nu_2(1) = 0.5772116186\ldots$; $\nu_3(1) = 0.5772156606\ldots$;
$\sigma_2(1) = 0.5772232685\ldots$; $\sigma_3(1) = 0.5772156685\ldots$.

As can be seen, $\nu_3(1)$ and $\sigma_3(1)$ are accurate to eight decimal places in approximating γ.

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