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The natural connectivity of colored random graphs

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ABSTRACT. The natural connectivity as a robustness measure of complex network has been proposed recently. It can be regarded as the average eigenvalue obtained from the graph spectrum. In this paper, we introduce an inhomogeneous random graph model, $G(n, \{c_i\}, \{p_i\})$, and investigate its natural connectivity. Binomial random graph $G\Big(n, \sum_{k=1}^{m} c_k^2 p_k\Big)$ is a tight approximation for $G(n, \{c_i\}, \{p_i\})$. Simulations are performed to validate our theoretical results.

1. INTRODUCTION

The classical approaches for determining robustness of networks stem from graph theory. For example, the vertex/edge connectivity of a graph is a fundamental measure of robustness of a network [3]. Recently, the concept of natural connectivity is proposed in [11, 16] as a novel spectral measure of robustness in networks. The natural connectivity is expressed as the weighted sum of closed walks of all lengths. The authors consider the redundancy of walks as the root of robustness, which ensures that the connection between vertices still remains possible in spite of damage to the network. The work [10, 17] analyze the natural connectivity in Erdős-Rényi random graph $G(n, p)$ [4] and show that the natural connectivity has acute discrimination in measuring the robustness of networks.

In this paper, we introduce a kind of inhomogeneous random graph model $G(n, \{c_i\}, \{p_i\})$, which we refer to as colored random graph, and study its robustness based on the natural connectivity. The natural connectivity of colored random graph can be tuned by assigning different color probabilities c_i as well as link probabilities p_i (see Section 2 for details). In addition, the natural connectivity result we derived in the present paper slightly improves that in [17] for $G(n, p)$ by a factor $n^{-\varepsilon}$. Simulations are provided to validate and illustrate our results.

The rest of this paper is organized as follows. Section 2 contains the definition of colored random graph model and some preliminaries for natural connectivity. The analytical results and simulation studies are given in Section 3 and 4, respectively. We conclude the paper in the final section.

2. MODEL AND PRELIMINARIES

In this section, we present some necessary preliminaries leading to the natural connectivity and colored random graphs.

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set $E \subseteq V \times V$. Let $|V| = n$ be the number of vertices. Let $A(G) = (a_{ij})_{n\times n}$ be the adjacency matrix of G, where $a_{ij} = a_{ji} = 1$ if $(i, j) \in E$, and $a_{ij} = a_{ji} = 0$ otherwise.

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Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A since it is real and symmetric. The set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called the spectrum of A (or G).

A weighted sum of numbers of closed walks is defined in [16] by

$$S = \sum_{k=0}^{\infty} n_k/k!, \tag{2.1}$$

where n_k is the number of closed walks of length k in G. Since $n_k = \sum_{i=1}^{n} \lambda_i^k$ [2], we have

$$S = \sum_{k=0}^{\infty} \sum_{i=1}^{n} \frac{\lambda_i^k}{k!} = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = \sum_{i=1}^{n} e^{\lambda_i}. \tag{2.2}$$

Note that (2.2) corresponds to the Estrada index of the graph [5, 12], which has been developed for the study of bipartivity [7] and subgraph centrality [6, 8]. The natural connectivity of G is then defined as

$$\bar{\lambda}(G) = \ln\left(\frac{S}{n}\right) = \ln\left(\frac{1}{n}\sum_{i=1}^{n} e^{\lambda_i}\right), \tag{2.3}$$

which corresponds to a kind of average eigenvalue of A since $\lambda_n \leq \bar{\lambda} \leq \lambda_1$.

Next we introduce our random graph model. Let m be a natural number and $c_i, p_i \in [0,1]$ for $i = 1, \cdots, m$. Suppose that $\sum_{i=1}^{m} c_i = 1$, and V is a set of n vertices. The colored random graph $G(n, \{c_i\}, \{p_i\})$ on V is defined as follows.

Let $\{1, 2, \cdots, m\}$ be m sorts of colors, and we consider a random coloring of the vertices in V by

$$f : V \to \{1, 2, \cdots, m\}. \tag{2.4}$$

For each vertex $v \in V$, we define $P(f(v) = i) = c_i$ and the coloring of a vertex is independent with that of other vertices. In other words, n vertices are assigned colors independently and identically distributed. For each pair of different vertices (v_i, v_j), an edge occurs with probability p_k if and only if $f(v_i) = f(v_j) = k$. There is no edges between v_i and v_j if $f(v_i) \neq f(v_j)$. We make a reference to the work [13], where the hyperbolicity of colored random graphs has been studied.

Clearly, the binomial random graph model $G(n, p)$ can be viewed as the special case of $m = 1$, i.e., $G(n, 1, p)$. It is straightforward to show that, for a pair of different vertices (v_i, v_j), the edge is present with probability $\sum_{k=1}^{m} c_k^2 p_k$. In the sequel, our analysis will based on the binomial random graph model $G\left(n, \sum_{k=1}^{m} c_k^2 p_k\right)$, and our simulation results in Section 4 implies that it is a sharp approximation for large n as far as the natural connectivity is concerned.

3. Main result

Our main result in this section is the following result.

Theorem 3.1. *For random graph* $G\left(n, \sum_{k=1}^{m} c_k^2 p_k\right)$ *with*

$$\frac{\ln n}{n} \ll \sum_{k=1}^{m} c_k^2 p_k < 1 - \frac{\ln n}{n}, \tag{3.5}$$

the natural connectivity is

$$\bar{\lambda}\Big(G\Big(n,\sum_{k=1}^{m}c_k^2p_k\Big)\Big)=n\sum_{k=1}^{m}c_k^2p_k-\ln n+o(1) \tag{3.6}$$

almost surely, as $n\to\infty$.

A key technique in the proof is a spectral density representation of random graph $G(n,p)$. It is shown [9] that the largest eigenvalue λ_1 of $G(n,p)$ is almost surely $(1+o(1))np$ provided that $np\gg\ln n$. Furthermore, Wigner's semicircular law [14, 15] says that the spectral density of $G(n,p)$ converges to the semicircular distribution

$$\rho(\lambda)=\begin{cases}\dfrac{2\sqrt{r^2-\lambda^2}}{\pi r^2}, & |\lambda|\le r\\ 0, & |\lambda|>r\end{cases} \tag{3.7}$$

as $n\to\infty$, where $r=2\sqrt{np(1-p)}$ is the radius of the bulk part of the spectrum.

For $\sum\limits_{k=1}^{m}c_k^2p_k\gg(\ln n)/n$ and $n\to\infty$, by continuous approximation for λ_i in (2.3), the natural connectivity of $G\Big(n,\sum\limits_{k=1}^{m}c_k^2p_k\Big)$ can be reformulated in the spectral density form

$$\begin{aligned}\bar{\lambda}\Big(G\Big(n,\sum_{k=1}^{m}c_k^2p_k\Big)\Big) &= \ln\Big(\int_{-r}^{r}\rho(\lambda)e^{\lambda}\mathrm{d}\lambda+\frac{e^{\lambda_1}}{n}\Big) \\ &= \ln\Big(\psi(1)+\frac{e^{n\sum_{k=1}^{m}c_k^2p_k}}{n}\Big) \\ &= n\sum_{k=1}^{m}c_k^2p_k-\ln n+\ln\left(1+\frac{n\psi(1)}{e^{n\sum_{k=1}^{m}c_k^2p_k}}\right)\end{aligned} \tag{3.8}$$

where $\psi(t)$ is the moment generating function of density $\rho(\lambda)$ and

$$\psi(1)=\int_{-r}^{r}\frac{2\sqrt{r^2-\lambda^2}}{\pi r^2}e^{\lambda}\mathrm{d}\lambda=\frac{2}{\pi}\int_0^{\pi}e^{r\cos\theta}\sin^2\theta\mathrm{d}\theta. \tag{3.9}$$

The following lemma can be proved by involving a modified Bessel function [1].

Lemma 3.1. ([17]) *The function*

$$g(p)=n\psi(1)/e^{np}\sim n\sqrt{\frac{2}{\pi}}\frac{e^{r-np}}{r^{3/2}} \tag{3.10}$$

is monotonically decreasing for $(\ln n)/n<p<1-(\ln n)/n$ *as* $n\to\infty$, *where* $\psi(1)$ *is given by* (3.9) *and* r *is defined in* (3.7).

Now we are on the stage to prove our main result.

Proof of Theorem 3.1. Let $p=p_c=(\ln n)/n$. Therefore, $1-p_c\to1$ as $n\to\infty$, and $r\sim2\sqrt{\ln n}$ from the definition in (3.7). By (3.10) we get

$$\begin{aligned}g(p_c) &\sim n\sqrt{\frac{2}{\pi}}\frac{e^{r-np_c}}{r^{3/2}}\sim n\sqrt{\frac{2}{\pi}}\cdot\frac{e^{2\sqrt{\ln n}-\ln n}}{(2\sqrt{\ln n})^{3/2}} \\ &= \frac{n}{2\sqrt{\pi}}\cdot\frac{e^{-\ln n+2\sqrt{\ln n}}}{(\ln n)^{3/4}}=\frac{e^{2\sqrt{\ln n}}}{2\sqrt{\pi}(\ln n)^{3/4}}\end{aligned} \tag{3.11}$$

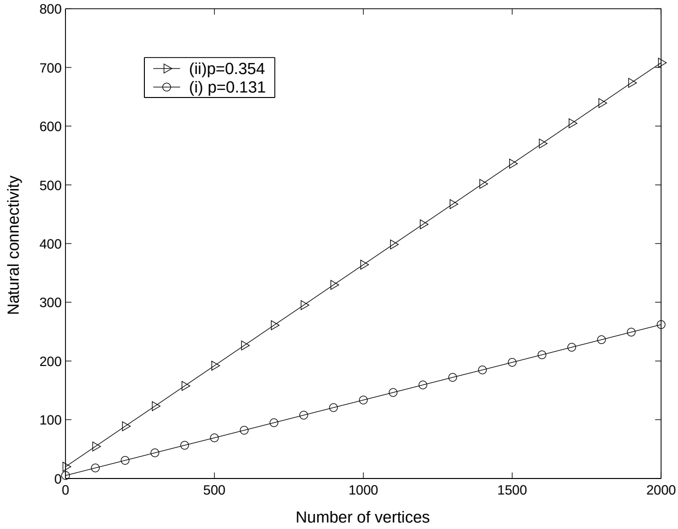

FIGURE 1. Natural connectivity $\bar{\lambda}$ of random graphs versus number of vertices n for different p. The lines represent $\bar{\lambda}(G(n,p))$ and the triangles and circles are for $\bar{\lambda}(G(n,\{c_i\},\{p_i\}))$. Each quantity is an average over 1000 realizations.

which tends to 0 as $n \to \infty$.
By Lemma 3.1, for $p_c \le p \le 1-p_c$, we have $g(p) \le g(p_c) \to 0$ as $n \to \infty$. Combining this with (3.8) and (3.10), we then conclude the proof of Theorem 3.1. $\square$

Notice that Theorem 3.1 improves the result in [17] (Theorem 3.3) by a factor of $n^{-\varepsilon}$.

4. SIMULATION STUDY

Firstly, we are interested in how well the approximation of $G\Big(n, \sum_{k=1}^{m} c_k^2 p_k\Big)$ behaves. We consider $m = 3$ colors. Let $c_1 = 0.1$, $c_2 = 0.3$, $c_3 = 0.6$.

(i) Take $p_1 = 0.5$, $p_2 = 0.6$, $p_3 = 0.2$. Then, we have $p = \sum_{k=1}^{3} c_k^2 p_k = 0.131$.

(ii) Take $p_1 = 0.3$, $p_2 = 0.7$, $p_3 = 0.8$. Then, we have $p = \sum_{k=1}^{3} c_k^2 p_k = 0.354$. For both cases, we simulate 1000 independent $G(n,p)$ and $G(n,\{c_i\},\{p_i\})$, and then we compute the average natural connectivity $\bar{\lambda}(G(n,p))$ and $\bar{\lambda}(G(n,\{c_i\},\{p_i\}))$ for different n.

Figure 1 shows the natural connectivities of $G(n,p)$ and $G(n,\{c_i\},\{p_i\})$. We observe that the two models agree well with each other.

Next, we illustrate the tunable natural connectivity for colored random graph $G(n,\{c_i\},\{p_i\})$. We consider the following two kinds of adjustment: color probabilities $\{c_i\}$ and link probabilities $\{p_i\}$.

(iii) Take $m = 2$ colors. Let $p_1 = 0.4$ and $p_2 = 0.9$. Therefore, the natural connectivity $\bar{\lambda}(G(n,\{c_i\},\{p_i\}))$ can be viewed as a function of color probability c_1 (with $c_2 = 1 - c_1$). In Figure 2, we plot the natural connectivity versus c_1 for $n = 1000$. We observe that the

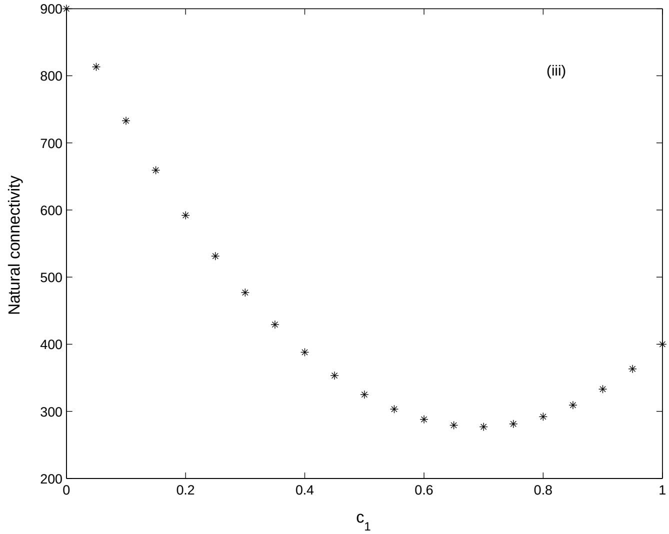

FIGURE 2. Natural connectivity $\bar{\lambda}(G(n,\{c_i\},\{p_i\}))$ versus color probability c_1 for $n = 1000$. Each quantity is an average over 1000 realizations.

natural connectivity changes with c_1 like a quadratic function and attains the minimum at $c_1 \approx 0.7$. This agrees with our theory since we have $\bar{\lambda} \sim (c_1^2 p_1 + (1-c_1)^2 p_2)n$ from (3.6).

(iv) Take $m = 3$ colors. Let $c_1 = 0.4$, $c_2 = 0.5$, $c_3 = 0.1$, $p_2 = 0.5$ and $p_3 = 0.8$. Therefore, the natural connectivity $\bar{\lambda}(G(n,\{c_i\},\{p_i\}))$ may be viewed as a function of link probability p_1. In Figure 3, we plot the natural connectivity versus p_1 for $n = 1000$. We observe from Figure 3 that the natural connectivity increases with p_1 linearly. This agrees with our theory since we have $\bar{\lambda} \sim \Big(\sum_{i=1}^{3} c_i^2 p_i\Big)n$ from (3.6). Compared with Figure 2, we know that the robustness of networks has a much more intricate relation with color probabilities $\{c_i\}$ than link probabilities $\{p_i\}$. This insight will be useful in the design and control of complex networks.

5. CONCLUSION

In this paper, we propose an inhomogeneous random graph model $G(n,\{c_i\},\ \{p_i\})$ through coloring each vertex of a graph and connecting vertices with the same colors with certain probabilities. Classical random graph $G(n,p)$ can be regarded as a special case of mono-colored random graph. We derive the natural connectivity of colored random graph $G\Big(n,\{c_i\},\{p_i\}\Big)$, which shows enough flexibility as for the parameters. This is of great theoretical and practical significance to the network robustness design and optimization. Extensive simulations are performed, which imply that binomial random graph $G\Big(n,\sum_{k=1}^{m} c_k^2 p_k\Big)$ is an excellent approximation for $G(n,\{c_i\},\{p_i\})$.

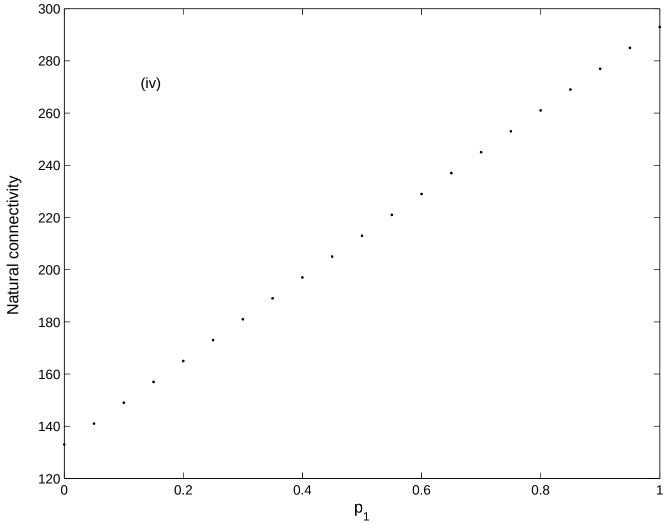

FIGURE 3. Natural connectivity $\bar{\lambda}(G(n, \{c_i\}, \{p_i\}))$ versus link probability p_1 for $n = 1000$. Each quantity is an average over 1000 realizations.

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