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Asymptotic expressions for the remainder term in the quadrature formula of Gauss-Jacobi type

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ABSTRACT. In this paper we have considered error analysis for a quadrature formula which is obtained by integration of linear positive operator. The asymptotic expressions for remainder term of Gauss-Jacobi type quadrature formula are also given.

1. INTRODUCTION

Many authors have considered error analysis for known and new quadrature rule. In [12] N. Ujević, N. Bilić obtained the following asymptotic expressions for error terms of the mid-point, trapezoid and Simpson's quadrature rule:

Denote by $\mathcal{F}=\left\{ f\in C^{\infty}[a,b] \left| \sup_{n\in\mathbb{N}} \left| \frac{f^{(n)}(a)}{f^{(n-1)}(a)} \right| \leq M<\infty \right.\right\}$.

Theorem 1.1. [12] *If $f\in\mathcal{F}$, then*

$$\int_a^b f(t)dt = f\left(\frac{a+b}{2}\right)(b-a)+\sum_{k=3}^{\infty}\frac{2^{k-1}-k}{2^{k-1}k!}f^{(k-1)}(a)(b-a)^k.$$

Theorem 1.2. [12] *If $f\in\mathcal{F}$, then*

$$\int_a^b f(t)dt = \frac{f(a)+f(b)}{2}(b-a)-\frac{1}{2}\sum_{k=3}^{\infty}\frac{k-2}{k!}f^{(k-1)}(a)(b-a)^k.$$

Theorem 1.3. [12] *If $f\in\mathcal{F}$, then*

$$\int_a^b f(t)dt=\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]+\frac{1}{3}\sum_{k=5}^{\infty}\frac{k+(k-6)\cdot 2^{k-3}}{2^{k-2}k!}f^{(k-1)}(a)(b-a)^k.$$

In this paper we will consider a class of quadrature formulas which are obtained by integration of linear positive operator. The asymptotic expressions for remainder term of these quadrature formulas are given. The above representations for remainder term of the mid-point and trapezoid rules are particular cases of our results.

The error analysis for quadrature formulas of Gauss type has occupied the attention of many authors ([5],[6]). An interesting asymptotical expansion of the remainder from Gauss type quadrature formulas was given by I. Gavrea in [7]. These results motivated us to give an asymptotical expansion of the remainder term from Gauss-Jacobi type quadrature formula.

In the next two sections we will recall some properties of Schurer-Stancu, respectively Gauss-Jacobi type quadrature formulas, which will be essentially used in the present paper and we will consider error analysis of these kind of quadrature formulas.

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2. The remainder term in the quadrature formula of Schurer-Stancu type

Let p be a given non-negative integer and let α, β be real parameters satisfying conditions $0 \leq \alpha \leq \beta$.

The Schurer-Stancu operators ([1]) $\tilde{S}_{m,p}^{(\alpha,\beta)} : C[0,1+p] \to C[0,1]$ are defined by

$$\left(\tilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{2.1}$$

where m is an positive integer and

$$\tilde{p}_{m,p}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$$

are the fundamental Schurer's polynomials.

Note that many approximation properties of operators (2.1) were investigated by D. Bărbosu in [3].

By integration the Schurer-Stancu approximation formula ([2], [3]),

$$f = \tilde{S}_{m,p}^{(\alpha,\beta)} f + \tilde{\mathcal{R}}_{m,p}^{\alpha,\beta} f,$$

was obtained the following quadrature formula

$$\int_0^1 f(x)dx = \sum_{k=0}^{m+p} \frac{1}{m+p+1} f\left(\frac{k+\alpha}{m+\beta}\right) + r_{m,p}^{(\alpha,\beta)}(f),$$

which it's called the Schurer-Stancu quadrature formula. Can be noted that the above quadrature formula ([9], [10]), in the case $\alpha = \beta = 0$ and $p \neq 0$ is the Schurer's quadrature formula, while for $\alpha = \beta = p = 0$ is the Bernstein's quadrature formula.

In what follows, we are dealing with the Schurer-Stancu quadrature fomula ([4]):

$$\int_0^1 f(x)dx = \sum_{k=0}^{m+p} \frac{1}{m+p+1} f\left(\frac{k+\alpha}{m+p+2\alpha}\right) + r_{m,p}^{(\alpha,2\alpha+p)}(f), \tag{2.2}$$

having the degree of exactness 1.

In [4] was proved that the remainder term of quadrature formula (2.2) has the following representation

$$r_{m,p}^{(\alpha,2\alpha+p)}(f) = \frac{(2\alpha-1)m + 2\alpha^2 + (2\alpha-1)p}{12(m+p+2\alpha)^2} f''(\xi),$$

where $f \in C^2[0,1]$ and $0 < \xi < 1$.

Lemma 2.1. [8] *If $-\infty < \alpha < \beta < +\infty$ and w is a weight function on (α, β) and*

$$\int_\alpha^\beta f(t)w(t)dt = \sum_{i=0}^m A_i f(x_i) + r_m[f], \; f \in L_w^1(\alpha,\beta),$$

then

$$W(x) = w\left(\alpha + (\beta-\alpha)\frac{x-a}{b-a}\right), \; x \in (a,b), \; -\infty < a < b < +\infty,$$

is a weight function on (a,b) and

$$\int_a^b F(x)W(x)dx = \frac{b-a}{\beta-\alpha} \sum_{i=0}^m A_i F\left(a + (b-a)\frac{x_i-\alpha}{\beta-\alpha}\right) + \mathcal{R}_m[F],$$

where $F \in L^1_w(a,b)$ *and*

$$\mathcal{R}_m[F] = \frac{b-a}{\beta-\alpha} r_m[\tilde{F}],\ \ \tilde{F}(t) = F\left(a + (b-a)\frac{t-\alpha}{\beta-\alpha}\right).$$

Using Lemma 2.1 the quadrature formula (2.2) can be written in the following form

$$\int_a^b f(x)dx = \sum_{k=0}^{m+p} \frac{b-a}{m+p+1} f\left(a + (b-a)\frac{k+\alpha}{m+2\alpha+p}\right) + \mathcal{R}_m[f], \tag{2.3}$$

where

$$\mathcal{R}_m[f] = (b-a)^3 f''(\xi) \cdot \frac{(2\alpha-1)m + 2\alpha^2 + (2\alpha-1)p}{12(m+p+2\alpha)^2},\ \ a < \xi < b. \tag{2.4}$$

In this section we will give an asymptotic expressions for remainder term of Schurer-Stancu type quadrature formula (2.3).

Theorem 2.4. *Let* $f \in C^\infty[a,b]$ *and* $\{f^{(i)}(a)\}_{i \geq 3}$ *is a bounded and monotonic sequence, then the remainder term of quadrature* (2.3) *has the following representation*

$$\mathcal{R}_m[f] = \sum_{i=3}^{\infty} c_i f^{(i-1)}(a),\ \ \textit{where} \tag{2.5}$$

$$c_i = \frac{(b-a)^i}{i!}\left[1 - i\sum_{k=0}^{m+p} \frac{1}{m+p+1}\left(\frac{k+\alpha}{m+2\alpha+p}\right)^{i-1}\right].$$

Proof. We define the function

$$\mathcal{R}(x) = \int_a^x f(t)dt - \sum_{k=0}^{m+p} \frac{x-a}{m+p+1} f\left(a + (x-a)\frac{k+\alpha}{m+2\alpha+p}\right).$$

By induction to show that for $i \in \mathbb{N}$, $i \geq 1$ we have

$$\begin{aligned}\mathcal{R}^{(i)}(x) = {} & f^{(i-1)}(x) - i\sum_{k=0}^{m+p} \frac{1}{m+p+1}\left(\frac{k+\alpha}{m+2\alpha+p}\right)^{i-1} f^{(i-1)}\left(a + (x-a)\frac{k+\alpha}{m+2\alpha+p}\right) \\ & - \sum_{k=0}^{m+p} \frac{x-a}{m+p+1}\left(\frac{k+\alpha}{m+2\alpha+p}\right)^{i} f^{(i)}\left(a + (x-a)\frac{k+\alpha}{m+2\alpha+p}\right).\end{aligned}$$

We have

$$\begin{aligned}\mathcal{R}(a) &= 0 \\ \mathcal{R}'(a) &= \left(1 - \sum_{k=0}^{m+p} \frac{1}{m+p+1}\right) f(a) = 0 \\ \mathcal{R}''(a) &= \left(1 - 2\sum_{k=0}^{m+p} \frac{1}{m+p+1} \cdot \frac{k+\alpha}{m+2\alpha+p}\right) f'(a) = 0 \\ \mathcal{R}^{(i)}(a) &= \left(1 - i\sum_{k=0}^{m+p} \frac{1}{m+p+1} \cdot \left(\frac{k+\alpha}{m+2\alpha+p}\right)^{i-1}\right) f^{(i-1)}(a),\ \ i \geq 3.\end{aligned}$$

If we now write the Taylor series

$$\mathcal{R}(x) = \sum_{i=0}^{\infty} \frac{\mathcal{R}^{(i)}(a)}{i!}(x-a)^i,$$

with the above data, we obtain

$$\mathcal{R}(x) = \sum_{i=3}^{\infty} c_i f^{(i-1)}(a), \text{ where}$$

$$c_i = \frac{(x-a)^i}{i!}\left[1 - i\sum_{k=0}^{m+p}\frac{1}{m+p+1}\left(\frac{k+\alpha}{m+2\alpha+p}\right)^{i-1}\right].$$

If we substitute $x = b$ in the above series then we get formula (2.5).

Now, we will prove that the series in (2.5) converges. In formula (2.2) we take $f(x) = x^{i-1}$, $i \geq 3$, and we find

$$1 - i\sum_{k=0}^{m+p}\frac{1}{m+p+1}\left(\frac{k+\alpha}{m+2\alpha+p}\right)^{i-1} = i(i-1)(i-2)\cdot\frac{(2\alpha-1)m+2\alpha^2+(2\alpha-1)p}{12(m+p+2\alpha)^2}\xi_i^{i-3},$$

where $0 < \xi_i < 1,\ i \geq 3$.

Using the above relation, the series (2.5) can be written

$$\mathcal{R}_m[f] = \frac{(2\alpha-1)m+2\alpha^2+(2\alpha-1)p}{12(m+p+2\alpha)^2}\cdot\sum_{i=3}^{\infty}\frac{(b-a)^i}{(i-3)!}\xi_i^{i-3}f^{(i-1)}(a).$$

Since $0 \leq \frac{1}{(i-3)!}(b-a)^i\xi_i^{i-3} \leq \frac{1}{(i-3)!}(b-a)^i$, and $\sum_{i=3}^{\infty}\frac{1}{(i-3)!}\cdot(b-a)^i$ is a convergent series, it follows that $\sum_{i=3}^{\infty}\frac{1}{(i-3)!}(b-a)^i\xi_i^{i-3}$ is a convergent series. Using the bellow theorem, it follows that the series in (2.5) converges.

Abel′ s Theorem. *If $\sum a_n$ is a convergent series and (b_n) is a monotonic and bounded sequence, then $\sum a_n b_n$ is a convergent series.* □

Remark 2.1. For $m = p = 0$, $\alpha = 1$, respectively $m = 1$ $p = 0$, $\alpha = 0$, we obtain the asymptotic expressions for the remainder term of the mid-point and trapezoid quadrature rule from Theorem 1.1, respectively Theorem 1.2.

3. The remainder term in the quadrature formula of Gauss-Jacobi type

By $J_m^{(\alpha,\beta)}$, where m is a nonnegative whole number and $\alpha, \beta > -1$, we denote the mth Jacobi polynomial. It is known that Jacobi polynomials with the same parameters α and β are orthogonal on $[-1,1]$ with respect to the weight function $\rho(x) = (1-x)^\alpha(1+x)^\beta$.

The quadrature formula of Gauss-Jacobi type generalized has the form ([10])

$$\int_a^b (b-x)^\alpha(x-a)^\beta f(x)dx = \sum_{k=0}^{m} B_{m,k}f(\gamma_k) + \mathcal{R}_m[f]. \tag{3.6}$$

The nodes γ_k , $k = \overline{0,m}$, which appear in (3.6) are given by

$$\gamma_k = \frac{b-a}{2}a_k + \frac{b+a}{2},$$

where a_k , $k=\overline{0,m}$ are the zeros of the Jacobi polynomial, $J_{m+1}^{(\alpha,\beta)}$ and

$$B_{m,k}=\frac{1}{2}\frac{(b-a)^{\alpha+\beta+1}(2m+\alpha+\beta+2)\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(m+1)!\Gamma(m+\alpha+\beta+2)J_m^{(\alpha,\beta)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(\alpha,\beta)}(x)\right]_{x=a_k}}.$$

For $f\in C^{2m+2}[a,b]$ the remainder term is given by

$$\mathcal{R}_m[f]=(b-a)^{2m+\alpha+\beta+3}\frac{f^{(2m+2)}(\xi)}{(2m+2)!}\cdot\frac{(m+1)!\Gamma(m+\alpha+2)\Gamma(m+\beta+2)\Gamma(m+\alpha+\beta+2)}{\Gamma(2m+\alpha+\beta+3)\Gamma(2m+\alpha+\beta+4)},$$

$$a<\xi<b\,.$$

Let $\alpha=\beta=0$. Then

$$\int_a^b f(x)dx=\frac{(b-a)}{m+1}\sum_{k=0}^{m}\frac{1}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f\left(\frac{b-a}{2}a_k+\frac{b+a}{2}\right)+\mathcal{R}_m[f],\quad(3.7)$$

where

$$\mathcal{R}_m[f]=\frac{(b-a)^{2m+3}(m+1)!^4}{(2m+3)(2m+2)!^3}f^{(2m+2)}(\xi).$$

If $a=-1$ and $b=1$ we obtain

$$\int_{-1}^{1} f(x)dx=\frac{2}{m+1}\sum_{k=0}^{m}\frac{1}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f(a_k)+\mathcal{R}_m[f],\qquad(3.8)$$

where

$$\mathcal{R}_m[f]=\frac{2^{2m+3}(m+1)!^4}{(2m+3)(2m+2)!^3}f^{(2m+2)}(\xi)\,,-1<\xi<1.$$

Remark 3.2. The quadrature formula (3.6) has the algebric degree of exactness $2m+1$.

If $f\in\mathcal{P}_{2m+1}$, then from (3.8) we have:

$$\frac{1}{m+1}\sum_{k=0}^{m}\frac{f(a_k)}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}=\frac{1}{2}\int_{-1}^{1}f(t)dt.\qquad(3.9)$$

If $f\in C^{2m+2}[a,b]$, then for any $x\in(a\ ,\ b]$ there is $\xi_x\in(a,x)$ such that

$$\int_a^x f(t)dt=\frac{(x-a)}{m+1}\sum_{k=0}^{m}\frac{1}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f\left(\frac{x-a}{2}a_k+\frac{x+a}{2}\right)+\mathcal{R}_m[f],\qquad(3.10)$$

where

$$\mathcal{R}_m[f]=\frac{(x-a)^{2m+3}(m+1)!^4}{(2m+3)(2m+2)!^3}f^{(2m+2)}(\xi_x).$$

Theorem 3.5. *If $f\in C^\infty[a,b]$ and $\{f^{(i)}(a)\}_{i\geq 2m+2}$ is a bounded and monotonic sequence, then the remainder term of quadrature formula (3.7) has the following representation*

$$\mathcal{R}_m[f]=\sum_{i=2m+3}^{\infty}c_if^{(i-1)}(a),\qquad(3.11)$$

where

$$c_i = \frac{(b-a)^i}{i!}\left\{1-\frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\right\}.$$

Proof. We define the function

$$\mathcal{R}(x)=\int_a^x f(t)dt-\frac{(x-a)}{m+1}\sum_{k=0}^{m}\frac{1}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f\left(\frac{x-a}{2}a_k+\frac{x+a}{2}\right).$$

By induction to show that for $i\in\mathbb{N},\ i\geq 1$ we have

$$\mathcal{R}^{(i)}(x)=f^{(i-1)}(x)-\frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f^{(i-1)}\left(\frac{x-a}{2}a_k+\frac{x+a}{2}\right)$$

$$-\frac{x-a}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{i}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f^{(i)}\left(\frac{x-a}{2}a_k+\frac{x+a}{2}\right).$$

For $i=\overline{1,2m+2}$, $\mathcal{R}^{(i)}(a)$ is given by

$$\mathcal{R}^{(i)}(a)=f^{(i-1)}(a)\left[1-\frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\right].$$

In formula (3.8) we take $f(x)=\left(\dfrac{x+1}{2}\right)^{i-1}$, $i=\overline{1,2m+2}$ and we find

$$1-\frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}=0,\text{namely } \mathcal{R}^{(i)}(a)=0,\ i=\overline{1,2m+2}.$$

For $i=2m+3$ we have

$$\mathcal{R}^{(2m+3)}(a)=f^{(2m+2)}(a)\left[1-\frac{2m+3}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{2m+2}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\right].$$

In formula (3.8) we take $f(x)=\left(\dfrac{x+1}{2}\right)^{2m+2}$ and we find

$$1-\frac{2m+3}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{2m+2}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}-\frac{(m+1)!^4}{(2m+2)!^2}=0,\text{namely}$$

$$\mathcal{R}^{(2m+3)}(a) = \frac{(m+1)!^4}{(2m+2)!^2} f^{(2m+2)}(a).$$

For $i \geq 2m+3$ we have

$$\mathcal{R}^{(i)}(a) = f^{(i-1)}(a) \left[1 - \frac{i}{m+1} \sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k) \frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}} \right].$$

If we now write the Taylor series

$$\mathcal{R}(x) = \sum_{i=0}^{\infty} \frac{\mathcal{R}^{(i)}(a)}{i!}(x-a)^i,$$

with the above data we have

$$\mathcal{R}(x) = \sum_{i=2m+3}^{\infty} \left[1 - \frac{i}{m+1} \sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k) \frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}} \right] f^{(i-1)}(a) \frac{(x-a)^i}{i!}.$$

If we substitute $x = b$ in the above series then we get the formula (3.11). Now, we want to show that the series in (3.11) converges. In formula (3.8) we take $f(x) = \left(\frac{x+1}{2}\right)^{i-1}$ and we find

$$1 - \frac{i}{m+1} \sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k) \frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}$$

$$= \frac{(m+1)!^4}{(2m+3)(2m+2)!^3} \cdot \frac{i!}{(i-2m-3)!} \left(\frac{\xi_i+1}{2}\right)^{i-2m-3}, \text{ where } -1 < \xi_i < 1.$$

Using the above relations, the series (3.11) can be written

$$\mathcal{R}[f] = \frac{(m+1)!^4}{(2m+3)(2m+2)!^3} \sum_{i=2m+3}^{\infty} \frac{1}{(i-2m-3)!} \left(\frac{\xi_i+1}{2}\right)^{i-2m-3} (b-a)^i f^{(i-1)}(a).$$

Since $\frac{1}{(i-2m-3)!} \left(\frac{\xi_i+1}{2}\right)^{i-2m-3} (b-a)^i \leq \frac{1}{(i-2m-3)!}(b-a)^i$, and $\sum_{i=2m+3}^{\infty} \frac{1}{(i-2m-3)!}(b-a)^i$ is a convergent series, it follows that $\sum_{i=2m+3}^{\infty} \frac{1}{(i-2m-3)!} \left(\frac{\xi_i+1}{2}\right)^{i-2m-3} (b-a)^i$ is a convergent series. Using Abel′ s Theorem, it follows that the series in (3.11) converges. $\square$

Theorem 3.6. *If $f \in C^{\infty}[a,b]$ and $\{f^{(i)}(b)\}_{i \geq 2m+2}$ is a bounded and monotonic sequence, then the remainder term of quadrature formula* (3.7) *has the following representation*

$$\mathcal{R}_m[f] = \sum_{i=2m+3}^{\infty} \tilde{c}_i f^{(i-1)}(b), \tag{3.12}$$

where

$$\tilde{c}_i = (-1)^{i-1}\frac{(b-a)^i}{i!}\left\{1-\frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\frac{1-a_k}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\right\}.$$

Proof. We define the function

$$\mathcal{R}(x) = \int_x^b f(t)dt - \frac{(b-x)}{m+1}\sum_{k=0}^{m}\frac{1}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f\left(\frac{b-x}{2}a_k+\frac{b+x}{2}\right).$$

By induction to show that for $i\in\mathbb{N},\ i\geq 1$ we have

$$\mathcal{R}^{(i)}(x) = -\left[f^{(i-1)}(x) - \frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{1-a_k}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}} f^{(i-1)}\left(\frac{b-x}{2}a_k+\frac{b+x}{2}\right)\right.$$
$$\left.-\frac{x-b}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{1-a_k}{2}\right)^{i}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\cdot f^{(i)}\left(\frac{b-x}{2}a_k+\frac{x+b}{2}\right)\right].$$

If we now write the Taylor series

$$\mathcal{R}(x) = \sum_{i=0}^{\infty}\frac{\mathcal{R}^{(i)}(b)}{i!}(x-b)^i,$$

with the above data we have

$$\mathcal{R}(x) = -\sum_{i=2m+3}^{\infty}\left[1-\frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{1-a_k}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\dfrac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\right] f^{(i-1)}(b)\frac{(x-b)^i}{i!}.$$

If we substitute $x=a$ in the above series then we get the formula (3.12). $\square$

Theorem 3.7. *Let $f\in C^{n+1}[a,b]$. Then for $n\geq 2m+3$ the remainder term has the following representation*

$$\mathcal{R}_m[f] = \sum_{i=2m+3}^{n} c_i f^{(i-1)}(a) + \frac{1}{n!}\int_a^b \mathcal{R}^{(n+1)}(t)(b-t)^n dt, \tag{3.13}$$

where

$$\mathcal{R}^{(i)}(t) = f^{(i-1)}(t) - \frac{i}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\dfrac{d}{dx}\left[J_{m+1}^{(0,0)}(x)\right]_{x=a_k}}\cdot f^{(i-1)}\left(\frac{t-a}{2}a_k+\frac{t+a}{2}\right)$$
$$-\frac{t-a}{m+1}\sum_{k=0}^{m}\frac{\left(\dfrac{a_k+1}{2}\right)^{i}}{J_m^{(0,0)}(a_k)\dfrac{d}{dx}\left[J_{m+1}^{(0,0)}(x)\right]_{x=a_k}}\cdot f^{(i)}\left(\frac{t-a}{2}a_k+\frac{t+a}{2}\right),\ \textit{for}\ i=\overline{1,n+1},$$

and

$$\mathcal{R}_m[f] = \sum_{i=2m+3}^{n} \tilde{c}_i f^{(i-1)}(b) + \frac{1}{n!}\int_a^b \tilde{\mathcal{R}}^{(n+1)}(t)(a-t)^n dt, \tag{3.14}$$

where

$$\tilde{\mathcal{R}}^{(i)}(t) = -\left[f^{(i-1)}(t) - \frac{i}{m+1}\sum_{k=0}^{m} \frac{\left(\frac{1-a_k}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(0,0)}(x)\right]_{x=a_k}} f^{(i-1)}\left(\frac{b-t}{2}a_k + \frac{b+t}{2}\right)\right.$$

$$\left. - \frac{t-b}{m+1}\sum_{k=0}^{m} \frac{\left(\frac{1-a_k}{2}\right)^{i}}{J_m^{(0,0)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(0,0)}(x)\right]_{x=a_k}} \cdot f^{(i)}\left(\frac{b-t}{2}a_k + \frac{t+b}{2}\right)\right], \text{ for } i = \overline{1, n+1}.$$

Proof. Let $\mathcal{R}(x)$ defined in the proof of Theorem 3.5. Using Taylor formula, we obtain

$$\mathcal{R}(x) = \sum_{i=0}^{n} \frac{\mathcal{R}^{(i)}(a)}{i!}(x-a)^i + \int_a^x \frac{\mathcal{R}^{(n+1)}(t)}{n!}(x-t)^n dt.$$

If we substitute $x = b$ in the above relation, we obtain the representation (3.13) of the remainder term. In a similar way we can obtain the relation (3.14). □

Theorem 3.8. *If $f \in C^\infty[a,b]$ and $\{f^{(i)}(a)\}_{i \geq 2m+2}$ is a bounded and monotonic sequence, then the remainder term of quadrature formula* (3.7) *has the following representation*

$$\mathcal{R}_m[f] = \frac{(b-a)^{2m+3}}{(2m+3)!} \cdot \frac{(m+1)!^4}{(2m+2)!^2} \cdot \left\{ f^{(2m+2)}(a) + \frac{b-a}{2} f^{(2m+3)}(a) \right\}$$

$$+ \sum_{i=2m+5}^{\infty} c_i f^{(i-1)}(a), \tag{3.15}$$

where

$$c_i = \frac{(b-a)^i}{i!}\left\{ 1 - \frac{i}{m+1}\sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{i-1}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}} \right\}.$$

Proof. Using the relation (3.11) we can write

$$\mathcal{R}[f] = c_{2m+3} f^{(2m+2)}(a) + c_{2m+4} f^{(2m+3)}(a) + \sum_{i=2m+5}^{\infty} c_i f^{(i-1)}(a), \tag{3.16}$$

where

$$c_{2m+3} = \frac{(b-a)^{2m+3}}{(2m+3)!}\left\{ 1 - \frac{2m+3}{m+1}\sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{2m+2}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}} \right\},$$

$$c_{2m+4} = \frac{(b-a)^{2m+4}}{(2m+4)!}\left\{ 1 - \frac{2m+4}{m+1}\sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{2m+3}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}} \right\}.$$

From the proof of Theorem 3.5 we have

$$1-\frac{2m+3}{m+1}\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{2m+2}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}=\frac{(m+1)!^4}{(2m+2)!^2}.$$

Therefore

$$c_{2m+3}=\frac{(b-a)^{2m+3}}{(2m+3)!}\cdot\frac{(m+1)!^4}{(2m+2)!^2}. \tag{3.17}$$

We will use the following properties of Jacobi polynomials (see [11]):

$$\begin{aligned} J_{2\nu}^{(\alpha,\alpha)}(x)&=\frac{\Gamma(2\nu+\alpha+1)\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)\Gamma(2\nu+1)}J_\nu^{(\alpha,-\frac12)}(2x^2-1),\\ J_{2\nu+1}^{(\alpha,\alpha)}(x)&=\frac{\Gamma(2\nu+\alpha+2)\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)\Gamma(2\nu+2)}xJ_\nu^{(\alpha,\frac12)}(2x^2-1).\end{aligned} \tag{3.18}$$

Also, for Jacobi polynomials the following formula holds

$$\frac{d}{dx}\left\{J_m^{(\alpha,\beta)}(x)\right\}=\frac12(m+\alpha+\beta+1)J_{m-1}^{(\alpha+1,\beta+1)}(x)\,. \tag{3.19}$$

Let $a_k,\ k=\overline{0,m}$ the zeros of $J_{m+1}^{(0,0)}$, the Jacobi polynomial. From [11] for a_k, $k=\overline{0,m}$ we have the relation:

$$a_k+a_{m-k}=0\,. \tag{3.20}$$

From (3.18) and (3.19) we have:

$$J_{2\nu}^{(0,0)}(a_k)\frac{d}{dx}\left[J_{2\nu+1}^{(0,0)}(x)\right]_{x=a_k}=(2\nu+1)J_\nu^{(0,-\frac12)}(2a_k^2-1)J_\nu^{(1,-\frac12)}(2a_k^2-1)\,, \tag{3.21}$$

$$J_{2\nu+1}^{(0,0)}(a_k)\frac{d}{dx}\left[J_{2\nu+2}^{(0,0)}(x)\right]_{x=a_k}=a_k^2(2\nu+3)J_\nu^{(0,\frac12)}(2a_k^2-1)J_\nu^{(1,\frac12)}(2a_k^2-1)\,. \tag{3.22}$$

From (3.20) , (3.21) and (3.22) we obtain

$$\sum_{k=0}^{m}\frac{a_k^{2m+3}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}=0,\ \text{therefore}$$

$$\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{2m+3}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}=\frac{1}{2^{2m+3}}\left[\sum_{k=0}^{m}\binom{2m+3}{2m+3}\frac{a_k^{2m+3}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\right.$$

$$\left.+\sum_{k=0}^{m}\sum_{i=0}^{2m+2}\binom{2m+3}{i}\frac{a_k^{i}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\right]$$

$$=\frac{1}{2^{2m+3}}\sum_{i=0}^{2m+2}\binom{2m+3}{i}\sum_{k=0}^{m}\frac{a_k^{i}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}\,.$$

From (3.8) with $f(x)=x^i$, $i=\overline{0,2m+1}$ we obtain

$$\sum_{k=0}^{m}\frac{a_k^{i}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}=\frac{m+1}{2}\cdot\frac{1+(-1)^i}{i+1} \tag{3.23}$$

and for $f(x) = x^{2m+2}$ we have

$$\sum_{k=0}^{m} \frac{a_k^{2m+2}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}} = \frac{m+1}{2m+3}\left[1 - \frac{2^{2m+2}(m+1)!^4}{(2m+2)!^2}\right]. \tag{3.24}$$

By using (3.23) and (3.24) follows

$$\sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{2m+3}}{J_m^{(0,0)}(a_k)\frac{d}{dt}\left[J_{m+1}^{(0,0)}(t)\right]_{t=a_k}}$$

$$= \frac{m+1}{2^{2m+3}}\left\{\sum_{k=0}^{m}\begin{pmatrix} 2m+3 \\ 2k \end{pmatrix}\cdot\frac{1}{2k+1} + 1 - \frac{2^{2m+2}(m+1)!^4}{(2m+2)!^2}\right\}$$

$$= \frac{m+1}{2^{2m+3}}\left\{\frac{2^{2m+3}}{2m+4} - \frac{2^{2m+2}(m+1)!^4}{(2m+2)!^2}\right\} = (m+1)\left\{\frac{1}{2m+4} - \frac{(m+1)!^4}{2(2m+2)!^2}\right\}.$$

Therefore

$$c_{2m+4} = \frac{(b-a)^{2m+4}}{(2m+4)!}\cdot\frac{(m+2)(m+1)!^4}{(2m+2)!^2}. \tag{3.25}$$

From relations (3.16), (3.17), (3.25) we obtain the relation (3.15). □

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