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On a generalization of Euler constant in connection to di-Gamma function

LASZLO BALOG

ABSTRACT. In this paper we study the sequences $\{x_n\}$, $\{y_n\}$ defined for each $n \geq 1$ by

$$x_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n}{a} + b\right), \tag{0.1}$$

and

$$y_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right), \tag{0.2}$$

where $a \in (0, +\infty)$ and $b \in \left[0, \frac{1}{2a}\right]$, in connection to Gamma and di-Gamma function.

Our results generalize some previous ones in [Berinde, V. *A new generalization of Euler's constant*, Creat. Math.Inform. **18** (2009), No. 2, 123–128] and [Sântămărian, A., *A generalization of Euler constant*, Mediamira, Cluj-Napoca, 2008] and are inspired from the paper [Mortici, C., *Improved convergence towards generalized Euler-Mascheroni constant*, Appl. Math. Comput., 2009, doi: 10.1016/j.amc.2009.10.039].

1. INTRODUCTION

The Euler-Mascheroni constant, γ, is the limit of the sequence $(H_n - \ln n)_{n\geq 1}$, where H_n is a harmonic number, so

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

$$\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n, \ \ n \geq 1, \tag{1.3}$$

is convergent, since $(\gamma_n)_{n\geq 1}$ is decreasing and bounded, with

$$0 < \gamma_n < 1, \ \ n \geq 1.$$

The sequence $(\gamma_n)_{n\geq 1}$ and the constant γ have numerous applications in many areas of mathematics and science in general, as analysis, theory of probability, physics, applied statistics, special functions, or number theory. But, the sequence given by (1.3) is slowly convergent to γ. To compute the value of Euler-Mascheroni constant it is required the study of some inequalities. Examples of such inequalities are mentioned below

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}, \tag{1.4}$$

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}, \tag{1.5}$$

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where $n \geq 1$. There are many attempts to improve the inequalities which approximate the value of γ, for the detailes see [6, 5, 3] and there in bibliography.

It is well known that the sequence

$$\left(\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\geq 1}$$

is convergent for all $a \in (0,\infty)$. Therefore, we can define the function $\gamma : (0,\infty) \to \mathbb{R}$ given by

$$\gamma(a)=\lim_{n\to\infty}\left(\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right) \tag{1.6}$$

It is easy to see that

$$\gamma(1)=\gamma.$$

Using the sequences with general terms given by

$$x_n=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n}{a},\ \ n\geq 1, \tag{1.7}$$

respectively

$$y_n=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a},\ \ n\geq 1 \tag{1.8}$$

in [7] are proved the following inequalities

a) $x_n < x_{n+1} < \gamma(a) < y_{n+1} < y_n$ for all $n \geq 1$
b) $0 < \frac{1}{a} - \ln\left(1+\frac{1}{a}\right) < \gamma(a) < \frac{1}{a}$ for all $a > 0$
c) $\frac{1}{2(n+a)} < \gamma(a) - x_n < \frac{1}{2(n+a-1)}$ for every $n \geq 1$
d) $\frac{1}{2(n+a)} < y_n - \gamma(a) < \frac{1}{2(n+a-1)}$ for every $n \geq 1$.

Much more, we have

$$\lim_{n\to\infty} n\left[\gamma(a)-x_n\right]=\frac{1}{2} \tag{1.9}$$

$$\lim_{n\to\infty} n\left[y_n-\gamma(a)\right]=\frac{1}{2} \tag{1.10}$$

$$\lim_{n\to\infty} n^2\left[z_n-\gamma(a)\right]=\frac{1}{6}, \tag{1.11}$$

where

$$z_n=\frac{x_n+y_n}{2}=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\sqrt{\frac{(a+n-1)(a+n)}{a}},\ \ n\geq 1.$$

The above equalities ensure that the order of convergence of $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ is the same as of the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$, while the order of convergence of $(z_n)_{n\geq 1}$ is the same as of the sequence $\left(\frac{1}{n^2}\right)_{n\geq 1}$.

2. A GENERALIZATION OF EULER-MASCHERONI CONSTANT

In this part of the paper, following the ideas from [2, 3, 5], we enrich the inequalities from above.

Recently, V. Berinde [3] introduced the sequences with general terms given by

$$x_n(a,b) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\left(\frac{a+n}{a} + b\right), \quad n \geq 1, \tag{2.12}$$

respectively

$$y_n(a,b) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right), \quad n \geq 1. \tag{2.13}$$

The sequences $\{x_n(a,b)\}_{n\geq 1}$ and $\{y_n(a,b)\}_{n\geq 1}$ are both convergent to $\gamma(a,b)$ for all $a > 0$ and $b \in \left[0, \frac{1}{2a}\right]$. We have the next results:

Theorem 2.1. *Let $\{x_n(a,b)\}_{n\geq 1}$ and $\{y_n(a,b)\}_{n\geq 1}$ be the sequences given by* (2.12), *respectively* (2.13). *The next inequalities*

a) $x_n(a,b) < x_{n+1}(a,b) < \gamma(a,b) < y_{n+1}(a,b) < y_n(a,b)$ *for every* $n \geq 1$;

b) $0 < \frac{1}{a} - \ln\left(1 + b + \frac{1}{a}\right) < \gamma(a,b) < \frac{1}{a} - \ln(1+b)$ *for all* $a > 0$;

hold.

Remark 2.1. For $b = 0$ we have

$$\gamma(a,0) = \gamma(a).$$

Remark 2.2. *For $a = 1$ and $b = 0$ we have*

$$\gamma(1,0) = \gamma(1) = \gamma.$$

Remark 2.3. For $a = 0$ and $b = \frac{1}{2}$ from $(y_n)_{n\geq 1}$ we have

$$\gamma\left(0, \frac{1}{2}\right) = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right)\right].$$

This limit was used by De Temple [4] to construct a sequence convergent to γ with the same order of convergence as the sequence $\left(\frac{1}{n^2}\right)_{n\geq 1}$.

Theorem 2.2 (Berinde [3]). *Let $a \in (0,\infty)$ and $(a_n)_{n\geq 1}$ given by*

$$a_n = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a}\right).$$

Then $(a_n)_{n\geq 1}$ is convergent to $\gamma(a)$ and the next inequlities

$$\gamma(a) < a_{n+1} < a_n, \quad n \geq 1 \tag{2.14}$$

hold.

The order of convergence to $\{x_n(a,b)\}_{n\geq 1}$ and $\{y_n(a,b)\}_{n\geq 1}$ is given in the next result

Theorem 2.3. *The following equalities*

$$\lim_{n\to\infty} n\left(\gamma(a,b) - x_n(a,b)\right) = \frac{1}{2} + ab, \tag{2.15}$$

$$\lim_{n\to\infty} n\left(y_n(a,b)-\gamma\left(a,b\right)\right)=\frac{1}{2}-ab \tag{2.16}$$

hold.

In that follows, we construct some sequences of type (2.12), respectively (2.13), to establish bounds for the γ function. A general class of these type of sequences where proposed by Mortici [5] and are

$$\mu_n\left(a,b,c\right)=\sum_{k=0}^{n-2}\frac{1}{a+k}+\frac{c}{a+n-1}-\ln\left(\frac{a+n-1}{a}+b\right),\ \ n\geq 1, \tag{2.17}$$

where a,b,c are real constants which fulfil some requirements.

We will construct $\{\alpha_n\left(a,b\right)\}_{n\geq 1}$ an increasing sequence, respectively $\{\beta_n\left(a,b\right)\}_{n\geq 1}$ a decreasing sequence, which converge to $\gamma\left(a,b\right)$ and

$$x_n\left(a,b\right)<\alpha_n\left(a,b\right)<\gamma\left(a,b\right)<\beta_n\left(a,b\right)<y_n\left(a,b\right),\ \ n\geq 1. \tag{2.18}$$

To construct the sequence $\{\beta_n\left(a,b\right)\}_{n\geq 1}$, we use a general form

$$\beta_n\left(a,b\right)=\sum_{i=0}^{n-1}\frac{1}{a+i}-\ln\left(\frac{a+n-1}{a}+b\right)+\frac{k\left(a,b\right)}{a+n-1},\ \ n\geq 1, \tag{2.19}$$

where the function $k\left(a,b\right)$ is such that we can obtain increasing and decreasing sequences which converge to $\gamma\left(a,b\right)$.

Remark 2.4. If $k\equiv 0$, then

$$\beta_n\left(a,b\right)=y_n\left(a,b\right),n\geq 1.$$

Remark 2.5. For $k\left(a,b\right)=-\dfrac{1}{2}$ we get the sequence studied in the paper of Sântămărian [8].

We consider $k\left(a,b\right)=\dfrac{1}{2}-ab$, where $a>0$ and $0\leq b\leq\frac{1}{2a}$. So, we have the sequence $\left(u_n\left(a,b\right)\right)_{n\geq 1}$ given by

$$u_n\left(a,b\right)=\sum_{k=0}^{n-1}\frac{1}{a+k}-\ln\left(\frac{a+n-1}{a}+b\right)+\frac{1}{a+n-1}\cdot\left(\frac{1}{2}-ab\right),\ \ n\geq 1, \tag{2.20}$$

Lemma 2.1. *The sequence* $\left(u_n\left(a,b\right)\right)_{n\geq 1}$ *given by* (2.20) *is decreasing.*

Remark 2.6. For $b=0$ we obtain

$$u_n=\frac{1}{a}+\frac{1}{a+1}+\ldots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}+\frac{1}{2\left(a+n\right)-2},\ \ n\geq 1. \tag{2.21}$$

The sequence $\left(u_n\right)_{n\geq 1}$ given by (2.21) is used to proove the inequalities

$$\frac{1}{2\left(n+a\right)}<y_n-\gamma\left(a\right)<\frac{1}{2\left(n+a-1\right)},\ \ n\geq 1,$$

where y_n is given by (1.8).

We consider $k\left(a,b\right)=-\frac{1}{2}+ab$, where $a>0$ and $0\leq b\leq\frac{1}{2a}$. So, we have the sequence $\left(v_n\left(a,b\right)\right)_{n\geq 1}$ given by

$$v_n\left(a,b\right)=\sum_{k=0}^{n-1}\frac{1}{a+k}-\ln\left(\frac{a+n-1}{a}+b\right)-\frac{1}{a+n-1}\cdot\left(\frac{1}{2}-ab\right),\ \ n\geq 1, \tag{2.22}$$

Lemma 2.2. *The sequence* $(v_n(a,b))_{n\geq 1}$ *given by* (2.22) *is increasing.*

Now, we can establish the next result

Theorem 2.4. *The sequences* $\{y_n(a,b)\}_{n\geq 1}$, $\{u_n(a,b)\}_{n\geq 1}$ *and* $\{v_n(a,b)\}_{n\geq 1}$ *with general terms given by* (2.13), (2.20), *respectively* (2.22), *are all convergent to* $\gamma(a,b)$. *Moreover, the inequalities*

$$v_n(a,b) < v_{n+1}(a,b) < \gamma(a,b) < u_n(a,b) < u_{n+1}(a,b), \quad n \geq 1, \tag{2.23}$$

$$v_n(a,b) < y_n(a,b) < u_n(a,b), \quad n \geq 1. \tag{2.24}$$

hold.

In order to complete the inequalities from (2.18), we construct an increasing sequence $\{\alpha_n(a,b)\}_{n\geq 1}$. To do that, we consider the general form

$$\alpha_n(a,b) = \sum_{i=0}^{n-1} \frac{1}{a+i} - \ln\left(\frac{a+n}{a}+b\right) + \frac{h(a,b)}{a+n}, \quad n \geq 1, \tag{2.25}$$

where the function $h(a,b)$ is such that we can obtain increasing and decreasing sequences, convergent to $\gamma(a,b)$.

Remark 2.7. *If* $h \equiv 0$, *then*

$$\alpha_n(a,b) = x_n(a,b), n \geq 1.$$

If we consider $h(a,b) = \frac{1}{2} - ab$ then we obtain the sequence $(w_n(a,b))_{n\geq 1}$ with the general term given by

$$w_n(a,b) = \sum_{k=0}^{n-1} \frac{1}{a+k} - \ln\left(\frac{a+n}{a}+b\right) + \frac{1}{a+n}\cdot\left(\frac{1}{2}-ab\right), \quad n \geq 1, \tag{2.26}$$

Lemma 2.3. *The sequence* $(w_n(a,b))_{n\geq 1}$ *given by* (2.26) *is increasing.*

Remark 2.8. *For* $b=0$ *we obtain a new sequence* $(w_n)_{n\geq 1}$ *with*

$$w_n = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\frac{a+n}{a} + \frac{1}{2(a+n)}, \quad n \geq 1.$$

This one was used to prove the inequlities

$$\frac{1}{2(n+a)} < \gamma(a) - x_n < \frac{1}{2(n+a-1)}$$

where x_n *is given by* (1.7).

Theorem 2.5. *The sequences* $\{x_n(a,b)\}_{n\geq 1}$ *and* $\{w_n(a,b)\}_{n\geq 1}$*with genereal terms given by* (2.12), *respectively* (2.26), *are both convergent to* $\gamma(a,b)$. *Moreover, the next inequalities*

$$w_n(a,b) < w_{n+1}(a,b) < \gamma(a,b), \quad n \geq 1, \tag{2.27}$$

$$x_n(a,b) < w_n(a,b), \quad n \geq 1 \tag{2.28}$$

hold.

From Theorem 2.4 and Theorem 2.5 we obtain that

$$x_n(a,b) < w_n(a,b) < \gamma(a,b) < y_n(a,b) < u_n(a,b), \quad n \geq 1. \tag{2.29}$$

3. Some generalizations of γ

Theorem 3.6. *Let $a \in (0, +\infty)$ and $b \in \left[0, \frac{1}{2a}\right]$. We consider the sequences $\{u_n\}$, $\{v_n\}$ given by*

$$u_n = \left(1 + \frac{1}{a+n+ab}\right)^{a+n+ab} \text{ for all } n \geq 1 \tag{3.30}$$

and

$$v_n = \left(1 + \frac{1}{a+n+ab}\right)^{a+n+ab+1} \text{ for all } n \geq 1. \tag{3.31}$$

Then
(i)

$$u_n < u_{n+1} < e < v_{n+1} < v_n, \ \forall n = 1, 2, \ldots \tag{3.32}$$

(ii)

$$\frac{1}{a+n+ab+1} < L(a,b,n) < \frac{1}{a+n+ab}, \quad \forall n = 1, 2, \ldots, \tag{3.33}$$

where

$$L(a,b,n) = \ln(a+n+ab+1) - \ln(a+n+ab) = \ln\left(1 + \frac{1}{a+n+ab}\right).$$

We remind that Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z > 0$$

and the digamma is its logarithmic derivative, that is

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

We note that

$$\Gamma'(1) = \psi(1) = -\gamma,$$
$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2,$$

where γ is the Euler's constant.

Theorem 3.7. *The function $\gamma : (0, \infty) \to (0, \infty)$ given by*

$$\gamma(a) = \lim_{n \to \infty} \left(\frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln \frac{a+n-1}{a}\right) \tag{3.34}$$

for any $a \in (0, \infty)$ is strictly decreasing.

Theorem 3.8. *Let $b \in \left[0, \frac{1}{2a}\right]$ be a real parameter. The function $\gamma_b : (0, \infty) \to (0, \infty)$ given by*

$$\gamma_b(a) = \lim_{n \to \infty} \left[\frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right)\right] \tag{3.35}$$

for all $a \in (0, \infty)$ is strictly decreasing.

Proof. The function γ_b is differentiable, so we have to prove that

$$\gamma_b'(a) < 0 \text{ for all } a > 0.$$

We consider the function $y_n : (0,\infty) \to (0,\infty)$ given by

$$y_{b,n}(a) = \frac{1}{a} + \frac{1}{a+1} + \ldots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right). \tag{3.36}$$

By differentiation with respect to a we obtain

$$y_{b,n}'(a) = -\frac{1}{a^2} - \frac{1}{(a+1)^2} - \ldots - \frac{1}{(a+n-1)^2} + \frac{n-1}{a(a+n-1+ab)}.$$

Thus, for any $a > 0$ and $n \in \mathbb{N} \setminus \{0,1\}$ we have

$$y_n'(a,b) = -\sum_{k=1}^{n} \frac{1}{(a+k-1)^2} + \frac{n-1}{a(a+n-1+ab)}. \tag{3.37}$$

Since, for any $a > 0$ and $n \in \mathbb{N} \setminus \{0,1\}$ the inequality

$$\frac{1}{(a+n-1)^2} < \frac{1}{(n-1)^2}$$

holds, it results that

$$\sum_{n=1}^{\infty} \frac{1}{(a+n-1)^2} < \sum_{n=1}^{\infty} \frac{1}{(n-1)^2}.$$

The series $\sum\limits_{n=1}^{\infty} \frac{1}{(n-1)^2}$ is convergent, hence the series $\sum\limits_{n=1}^{\infty} \frac{1}{(a+n-1)^2}$ is uniformly convergent for all $a > 0$.

The sequence $(a_n)_{n\geq 1}$ defined by

$$a_n = \frac{n-1}{a(a+n-1+ab)}, \forall n = 1, 2, \ldots$$

is uniformly convergent for all $a > 0$ and $b \in \left[0, \frac{1}{2a}\right]$.

Therefore, $y_{b,n}'(a)$ is uniformly convergent on $(0,\infty)$ and γ_b is derivable on $(0,\infty)$. Moreover, we have

$$\gamma_b'(a) = \lim_{n\to\infty} y_{b,n}'(a) \text{ for every } a > 0, b \in \left[0, \frac{1}{2a}\right]$$

and

$$\gamma_b'(a) < -\frac{1}{a} < 0, \quad \forall a > 0, \quad b \in \left[0, \frac{1}{2a}\right]. \tag{3.38}$$

□

Remark that $\gamma_b(a) = \gamma(a,b)$ for all $a > 0$ and $b \in \left[0, \frac{1}{2a}\right]$.

In that follow, we make some estimations for

$$D(n,a,b) = y_n - \gamma(a,b), \quad \forall n = 1, 2, \ldots,$$

where $\{y_n\}$ is the sequence given by

$$y_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + b\right), \tag{3.39}$$

for any $n = 1, 2, \ldots$.

Theorem 3.9. *Let $a \in (0, \infty)$. Then*

$$\gamma(a, b) = \ln a - \psi(a), \tag{3.40}$$

where $\gamma(a, b)$ is the limit of sequence given by (3.39) *and ψ is the digamma function, i.e.,*

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \ \ \forall x > 0.$$

Proof. We prove these inequalities using the logarithmic derivative of the gamma function. First, we have

$$\begin{aligned}
\gamma(a, b) &= \lim_{n\to\infty}\left(\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n} - \ln\left(\frac{a+n+ab}{a}\right)\right) \\
&= \frac{1}{a} + \lim_{n\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right) \\
&+ \lim_{n\to\infty}\left(\ln n - \ln\frac{n+a+ab}{a}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{a+n}\right) \\
&= \frac{1}{a} + \gamma(1) + \ln a - a \cdot \sum_{n=1}^{\infty}\frac{1}{n(a+n)} \\
&= \frac{1}{a} + \ln a - \psi(1+a) = \ln a - \psi(a).
\end{aligned}$$

In conclusion, we have

$$\gamma(a, b) = \ln a - \psi(a).$$

□

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North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare, Romania
E-mail address: balog_58@yahoo.com