CREAT. MATH. INFORM.
**21** (2012), No. 1, 21 - 26

**Online version at `http://creative-mathematics.ubm.ro/`**
**Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X**

# A new homogeneous inequality and a few of its applications

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ABSTRACT. An original homogeneous inequality, written for 2 sets of n variables, represents the starting point of the paper. This inequality is the key to the other theoretical results, and could be considered as a simple, but new and powerful mathematical tool.

## 1. INTRODUCTION

Every year new problems enrich the large chapter of inequalities, and many of them are proposed in different math contests and olympiads. Proving inequalities can be often very difficult. Frequently, some famous inequalities are cited and applied without proof in a certain solution.

The main result of this article allows us to obtain some important inequalities, which are usually considered as consequences of Hölder's inequality, Chebyshev's inequality and Cauchy-Schwarz's inequality. But for proving the theoretical results emphasized in the paper I do not use any classical inequality.

## 2. AN IMPORTANT INEQUALITY

**Theorem 2.1.** *For $a_i \geq 0, b_i > 0, i = 1, 2, \ldots, n,\ n \in \mathbb{N}, n \geq 1$, and $k > 0$, the following inequality holds*

$$\frac{a_1^{k+1}}{b_1^k} + \frac{a_2^{k+1}}{b_2^k} + \ldots + \frac{a_n^{k+1}}{b_n^k} \geq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \left( \frac{a_1^k}{b_1^{k-1}} + \frac{a_2^k}{b_2^{k-1}} + \ldots + \frac{a_n^k}{b_n^{k-1}} \right),$$

*or in the simplified form*

$$\sum_{i=1}^{n} \frac{a_i^{k+1}}{b_i^k} \geq \frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i} \cdot \sum_{i=1}^{n} \frac{a_i^k}{b_i^{k-1}} \tag{2.1}$$

*Proof.* If $a_1 + a_2 + ... + a_n = 0$, then $a_1 = a_2 = ... = a_n = 0$, and the inequality is "trivial". For $a_1 + a_2 + ... + a_n > 0$, because (2.1) is homogenious in $a_1, a_2, ..., a_n$, we obtain by normalization that $a_1 + a_2 + ... + a_n = 1$.

Similarly, because (2.1) is homogenious in $b_1, b_2, ..., b_n$ and $b_1 + b_2 + ... + b_n > 0$, we obtain by normalization that $b_1 + b_2 + ... + b_n = 1$.

Therefore, without loss of generality, we can suppose that

$$a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n.$$

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Received: 31.10.2010. In revised form: 30.06.2011. Accepted: 30.11.2011
2010 *Mathematics Subject Classification.* 97H30.
Key words and phrases. *Homogeneous inequality, Cauchy-Schwarz inequality, Hölder's inequality, Chebyshev's inequality.*

The initial inequality becomes

$$\frac{{a_1}^{k+1}}{b_1^k}+\frac{a_2^{k+1}}{b_2^k}+\ldots+\frac{{a_n}^{k+1}}{b_n^k}\geq\frac{{a_1}^k}{b_1^{k-1}}+\frac{{a_2}^k}{b_2^{k-1}}+\ldots+\frac{{a_n}^k}{b_n^{k-1}}.$$

Therefore

$$\left(\frac{a_1^{k+1}}{b_1^k}-\frac{a_1^k}{b_1^{k-1}}\right)+\left(\frac{a_2^{k+1}}{b_2^k}-\frac{a_2^k}{b_2^{k-1}}\right)+\ldots+\left(\frac{a_n^{k+1}}{b_n^k}-\frac{a_n^k}{b_n^{k-1}}\right)\geq 0.$$

This inequality is equivalent to

$$\left(\frac{a_1}{b_1}\right)^k\cdot(a_1-b_1)+\left(\frac{a_2}{b_2}\right)^k\cdot(a_2-b_2)+\ldots+\left(\frac{a_n}{b_n}\right)^k\cdot(a_n-b_n)\geq 0.$$

There is no difficulty to prove that for $a\geq 0, b>0$ and $k>0$, the following inequality holds

$$\left(\frac{a}{b}\right)^k\cdot(a-b)\geq a-b.$$

Using this inequality for $a_i\geq 0, b_i>0, i=1,2,\ldots,n,\ n\in\mathbb{N}, n\geq 1$ and $k>0$, and adding all these $n$ inequalities, which are obtained for each pair $(a_i,b_i)$, we get the desired inequality. □

## 3. Some interesting results

**Corollary 3.1.** *For $a_i\geq 0, b_i>0, i=1,2,\ldots,n$, and $k,n\in\mathbb{N}$, with $n,k\geq 1$, the following inequality holds*

$$\frac{{a_1}^{k+1}}{b_1^k}+\frac{a_2^{k+1}}{b_2^k}+\ldots+\frac{{a_n}^{k+1}}{b_n^k}\geq\frac{(a_1+\ldots+a_n)^{k+1}}{(b_1+\ldots+b_n)^k}. \tag{3.2}$$

*Proof.* Using the inequality 2.1 (see the steps [1], [2], ...,[p],...,[k]), we get:

$$\frac{{a_1}^{k+1}}{b_1^k}+\ldots+\frac{{a_n}^{k+1}}{b_n^k}\geq$$

$$\stackrel{[1]}{\geq}\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\cdot\left(\frac{{a_1}^k}{b_1^{k-1}}+\ldots+\frac{{a_n}^k}{b_n^{k-1}}\right)\geq$$

$$\stackrel{[2]}{\geq}\left(\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\right)^2\cdot\left(\frac{{a_1}^{k-1}}{b_1^{k-2}}+\ldots+\frac{{a_n}^{k-1}}{b_n^{k-2}}\right)\geq$$

$$\ldots$$

$$\stackrel{[p]}{\geq}\left(\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\right)^p\cdot\left(\frac{{a_1}^{k-p+1}}{b_1^{k-p}}+\ldots+\frac{{a_n}^{k-p+1}}{b_n^{k-p}}\right)\geq$$

$$\ldots$$

$$\stackrel{[k]}{\geq}\left(\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\right)^k\cdot\left(\frac{a_1^1}{b_1^0}+\ldots+\frac{a_n^1}{b_n^0}\right)=\frac{(a_1+\ldots+a_n)^{k+1}}{(b_1+\ldots+b_n)^k}$$

Therefore, the inequality (3.2) is proved. □

**Remark 3.1.** It is easy to prove that if $k$ is an odd natural number, then the inequality (3.2) holds for any $a_i\in\mathbb{R}$.

**Corollary 3.2.** *For:* $a_i \in \mathbb{R}, b_i > 0, i = 1, 2, ..., n,$ *and* $n \in \mathbb{N}, n \geq 1,$ *the following inequality holds*

$$\frac{{a_1}^2}{b_1} + \frac{a_2^2}{b_2} + \ldots + \frac{{a_n}^2}{b_n} \geq \frac{(a_1 + \ldots + a_n)^2}{b_1 + \ldots + b_n} \tag{3.3}$$

*Proof.* We obtain this result using the inequality (3.2), and the last remark, for the particular case $k = 1$. □

**Lemma 3.1.** *For* $x_i \geq 0, i = 1, 2, ..., n,$ *and* $n \in \mathbb{N}, n \geq 1,$ *the following inequality holds*

$$n^{k-2} \cdot (x_1 + x_2 + \ldots + x_n) \geq (\sqrt[k-1]{x_1} + \sqrt[k-1]{x_2} + \ldots + \sqrt[k-1]{x_n})^{k-1}. \tag{3.4}$$

*Proof.* For proving the inequality (3.4), let us notice that

$$x_1 + x_2 + \ldots + x_n$$

is equal to

$$\frac{\left(\sqrt[k-1]{x_1}\right)^{k-1}}{1^{k-2}} + \frac{\left(\sqrt[k-1]{x_2}\right)^{k-1}}{1^{k-2}} + \ldots + \frac{\left(\sqrt[k-1]{x_n}\right)^{k-1}}{1^{k-2}}.$$

Using (3.2), we obtain that this value is greater or equal to

$$\frac{(\sqrt[k-1]{x_1} + \sqrt[k-1]{x_2} + \ldots + \sqrt[k-1]{x_n})^{k-1}}{(\underbrace{1 + 1 + \ldots + 1}_{n-times})^{k-2}} = \frac{1}{n^{k-2}} \cdot (\sqrt[k-1]{x_1} + \sqrt[k-1]{x_2} + \ldots + \sqrt[k-1]{x_n})^{k-1}.$$

Therefore

$$x_1 + x_2 + \ldots + x_n \geq \frac{1}{n^{k-2}} \cdot (\sqrt[k-1]{x_1} + \sqrt[k-1]{x_2} + \ldots + \sqrt[k-1]{x_n})^{k-1},$$

which is equivalent to (3.4). □

**Corollary 3.3.** *For* $a_i \geq 0, b_i > 0, i = 1, 2, ..., n$ *and* $n, k \in \mathbb{N},$ *with* $n \geq 1,\ k \geq 2,$ *the following inequality holds*

$$\frac{{a_1}^k}{b_1} + \frac{{a_2}^k}{b_2} + \ldots + \frac{{a_n}^k}{b_n} \geq \frac{(a_1 + a_2 + \ldots + a_n)^k}{n^{k-2} \cdot (b_1 + b_2 + \ldots + b_n)} \tag{3.5}$$

*Proof.* For $k = 2$, the inequality (3.5) reduces to the inequality (3.3).
Let us consider that $k \in \mathbb{N}, k \geq 3$. In this case, the left hand side of the inequality (3.5) is equal to

$$\frac{{a_1}^k}{(\sqrt[k-1]{b_1})^{k-1}} + \frac{{a_2}^k}{(\sqrt[k-1]{b_2})^{k-1}} + \ldots + \frac{{a_n}^k}{(\sqrt[k-1]{b_n})^{k-1}}.$$

Using the inequality (3.2), we get that this value is greater or equal to

$$\frac{(a_1 + a_2 + \ldots + a_n)^k}{(\sqrt[k-1]{b_1} + \sqrt[k-1]{b_2} + \ldots + \sqrt[k-1]{b_n})^{k-1}}.$$

But, from (3.4), we get

$$(\sqrt[k-1]{b_1} + \sqrt[k-1]{b_2} + \ldots + \sqrt[k-1]{b_n})^{k-1} \leq n^{k-2} \cdot (b_1 + b_2 + \ldots + b_n).$$

Therefore

$$\frac{(a_1 + a_2 + \ldots + a_n)^k}{(\sqrt[k-1]{b_1} + \sqrt[k-1]{b_2} + \ldots + \sqrt[k-1]{b_n})^{k-1}} \geq \frac{(a_1 + a_2 + \ldots + a_n)^k}{n^{k-2} \cdot (b_1 + b_2 + \ldots + b_n)},$$

and the inequality (3.5) is proved. □

**Corollary 3.4.** *For $a_i \geq 0, i = 1, 2, ..., n,$ and $n, k \in \mathbb{N}$, with $n, k \geq 2$, then the following inequality holds*

$$\sqrt[k]{\frac{a_1^k + a_2^k + \ldots + a_n^k}{n}} \geq \frac{a_1 + a_2 + \ldots + a_n}{n} \tag{3.6}$$

*Proof.* Using (3.2), for $b_1 = b_2 = ... = b_n = 1$, and $k \to (k-1)$, the inequality becomes

$$a_1^k + a_2^k + \ldots + a_n^k \geq \frac{(a_1 + a_2 + \ldots + a_n)^k}{n^{k-1}},$$

which is equivalent to

$$\frac{a_1^k + a_2^k + \ldots + a_n^k}{n} \geq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^k.$$

Therefore, using this inequality we get (3.6) immediately. □

**Corollary 3.5.** *For $a_i \geq 0, i = 1, 2, \ldots, n,$ and $n \in \mathbb{N}, n \geq 1,$ and $k > 0$, the following inequality holds*

$$a_1^{k+1} + a_2^{k+1} + \ldots + a_n^{k+1} \geq \frac{a_1 + a_2 + \ldots + a_n}{n} \cdot (a_1^k + a_2^k + \ldots + a_n^k) \tag{3.7}$$

*Proof.* Using the main inequality (2.1), for $b_1 = b_2 = ... = b_n = 1$, we get the inequality (3.7) immediately. □

## 4. Two beautiful generalizations of Nesbitt's inequality

A well known result, which is called Nesbitt's inequality, has the following statement.

**J. NESBITT's inequality (1903):** For $a, b, c > 0$, the following inequality holds

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \tag{4.8}$$

My examples consist in two beautiful generalizations of this inequality. For both generalizations, in the particular case $k = 1$, we obtain Nesbitt's inequality.

**Example 4.1.** For $a, b, c > 0$, and $k \in \mathbb{N}, k \geq 1$, the following inequality holds

$$\frac{a}{(b+c)^k} + \frac{b}{(c+a)^k} + \frac{c}{(a+b)^k} \geq \left(\frac{3}{2}\right)^k \frac{1}{(a+b+c)^{k-1}} \tag{4.9}$$

*Proof.* The left hand side of the inequality is equal to

$$\frac{a^{k+1}}{[a(b+c)]^k} + \frac{b^{k+1}}{[b(c+a)]^k} + \frac{c^{k+1}}{[c(a+b)]^k}$$

Using (3.2), we have that this value is greater or equal to

$$\frac{(a+b+c)^{k+1}}{[a(b+c) + b(c+a) + c(a+b)]^k},$$

which can be rewritten as

$$\frac{(a+b+c)^{k+1}}{2^k} \cdot \left(\frac{1}{ab+bc+ca}\right)^k.$$

Taking into account the "trivial" inequality

$$(a+b+c)^2 \geq 3(ab+bc+ca) > 0,$$

which implies that

$$\frac{1}{ab+bc+ca} \geq \frac{3}{(a+b+c)^2},$$

this value is greater or qual to

$$\frac{(a+b+c)^{k+1}}{2^k} \cdot \left[\frac{3}{(a+b+c)^2}\right]^k = \left(\frac{3}{2}\right)^k \cdot \frac{1}{(a+b+c)^{k-1}},$$

and the inequality (4.9) is proved. □

**Remark 4.2.** For $k = 2$ in (4.9), we get a well known inequality, whose author is Darij Grinberg (see the problem no. 7, from [4]).

**Example 4.2.** For $a, b, c > 0$, and $k \in \mathbb{N}, k \geq 1$, the following inequality holds

$$\frac{a^k}{b+c} + \frac{b^k}{c+a} + \frac{c^k}{a+b} \geq \frac{a^{k-1} + b^{k-1} + c^{k-1}}{2} \tag{4.10}$$

*Proof.* Setting $a + b + c = s$, the left hand side of the inequality is equal to

$$\frac{(a^{k-1})^2}{a^{k-2}(s-a)} + \frac{(b^{k-1})^2}{b^{k-2}(s-b)} + \frac{(c^{k-1})^2}{c^{k-2}(s-c)}.$$

Using (3.3), we have that this value is greater or equal to

$$\frac{(a^{k-1} + b^{k-1} + c^{k-1})^2}{a^{k-2}(s-a) + b^{k-2}(s-b) + c^{k-2}(s-c)},$$

which is equal to

$$\frac{(a^{k-1} + b^{k-1} + c^{k-1})^2}{s(a^{k-2} + b^{k-2} + c^{k-2}) - (a^{k-1} + b^{k-1} + c^{k-1})}.$$

Using (3.7), we obtain

$$a^{k-1} + b^{k-1} + c^{k-1} \geq \frac{a+b+c}{3} \cdot (a^{k-2} + b^{k-2} + c^{k-2}),$$

which implies that

$$3(a^{k-1} + b^{k-1} + c^{k-1}) \geq s(a^{k-2} + b^{k-2} + c^{k-2}).$$

Therefore the proof is finished, because from this inequality we get

$$\frac{(a^{k-1} + b^{k-1} + c^{k-1})^2}{s(a^{k-2} + b^{k-2} + c^{k-2}) - (a^{k-1} + b^{k-1} + c^{k-1})} \geq \frac{a^{k-1} + b^{k-1} + c^{k-1}}{2}.$$

□

**Remark 4.3.** Using the same steps, we can prove the following beautiful generalization of the inequality (4.10)

For $a_1, a_2, \ldots, a_n > 0, s = a_1 + a_2 + \ldots + a_n$, and $n, k \in N, n \geq 2, k \geq 1$, the following inequality holds:

$$\frac{a_1^k}{s-a_1} + \frac{a_2^k}{s-a_2} + \cdots + \frac{a_n^k}{s-a_n} \geq \frac{a_1^{k-1} + a_2^{k-1} + \cdots + a_n^{k-1}}{n-1}.$$

**Remark 4.4.** The inequality (4.10) is one of the problems proposed in [5] (see the problem no. 69, Chapter 7 - Inequalities), and the solution proposed by the author is different than our solution.

## 5. CONCLUSIONS

The inequality (2.1) represents the key to the other theoretical results. But, many authors use some classical inequalities for proving the theoretical results emphasized in this article.

For example, in [1], the author uses Hölder's inequality for proving the content of Corollary 3.1.

The content of Corollary 3.2 was proposed by Titu Andreescu, in "Revista de Matematică din Timişoara", in 1979, for the particular case $n = 2$ (see [2]). After that, using this starting point, the author proves the general case. Let us notice that the inequality (3.3) can be immediately proved using Cauchy - Schwarz' s inequality. The inequality (3.3) is known as Titu Andreescu's inequality, or $T_2$ $'s$ lemma, or Cauchy-Schwarz's inequality in Engel's form.

The inequality (3.6), which is contained in Corollary 3.4, is the inequality between the power mean, with exponent $k$, and the arithmetic mean, for the non-negative real numbers $a_1, a_2, ..., a_n$. In [3], the author uses this inequality for proving Corollary 3.3.

And finally, Corollary 3.5 is often considered as a direct consequence of Chebyshev's inequality.

We can conclude that the article provides to the reader some valuable results which can be used to solve a lot of difficult inequalities.

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