CREAT. MATH. INFORM.
21 (2012), No. 1, 35 - 40

Online version at `http://creative-mathematics.ubm.ro/`
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Blending surfaces generated using the Bernstein operator

MARIUS BIROU

ABSTRACT. In this paper we construct blending surfaces using the univariate Bernstein operator. The surfaces have the properties that they stay on a curve (the border of the surfaces domain) and have a fixed height in a point from the domain. The surfaces are generated using a curve network, instead of the control points from the case of classical Bezier surfaces. We study the monotonicity and we give conditions to obtain concave surfaces.

1. INTRODUCTION

The blending surfaces are surfaces which contain some given contours (curves, segments, points). They have been created by Coons S.A. [3]. In some previous papers (see [1, 2] and references therein), there were constructed blending surfaces that stay on a rectangle or a triangle (the border of the surfaces domain) and having a fixed height in the point (0,0). In this paper we use the univariate Bernstein polynomial to get the surfaces which stay on a curve (the surfaces domain are bounded by this curve). We assume that the point $(0,0)$ belongs to the surfaces domains and we fix the height of the surfaces in this point. We construct two families of surfaces for which we study the monotonicity and we give conditions to obtain concave surfaces. These surfaces can be used in civil engineering (roof surfaces) or in Computer Aided Geometric Design (CAGD).

2. PRELIMINARIES

The univariate Bernstein polynomial of a function $f:[0,1]\to\mathbb{R}$ is given by

$$(B_n f)(t)=\sum_{j=0}^{n} b_{jn}(t) f\left(\frac{j}{n}\right), \tag{2.1}$$

where the functions b_{jn} are given by formula

$$b_{jn}(t)=\binom{n}{j} t^j (1-t)^{n-j}$$

for $j=0,...,n$. It has the interpolation properties

$$(B_n f)(0)=f(0),\ (B_n f)(1)=f(1)$$

and the shape properties given by the following two theorems (see [7])

Theorem 2.1. *If the function $f:[0,1]\to R$ is increasing (decreasing) then the function $B_n f$ is increasing (decreasing).*

Theorem 2.2. *If the function $f:[0,1]\to R$ is convex (concave) then the function $B_n f$ is convex (concave).*

Received: 31.10.2010. In revised form: 30.06.2011. Accepted: 30.11.2011
2010 *Mathematics Subject Classification.* 65D17, 41A36, 41A63.
Key words and phrases. *Bernstein polynomial, surfaces, monotonicity, concavity .*

Next we give some definitions and remarks about the monotonicity and convexity of the bivariate functions (see [5, 6, 4]).

Definition 2.1. The bivariate function $G : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$ is increasing (decreasing) in a direction $d = (d_1, d_2) \in \mathbb{R}^2$ if and only if

$$G(x + \lambda d_1, y + \lambda d_2) \geq (\leq) G(x, y),$$

for every $(x, y) \in A$ and every $\lambda > 0$.

Remark 2.1. If G is a C^1 function on the set A we have that the function G is increasing (decreasing) in the direction $d = (d_1, d_2)$ if

$$D_d G \geq (\geq 0)$$

on A, where $D_d G$ is the first order directional derivative in the direction $d = (d_1, d_2)$ of the function G, i.e.

$$D_d G = d_1 G_x + d_2 G_y.$$

Definition 2.2. The bivariate function $G : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^2$ is convex (concave) on the convex set A if and only if

$$G(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \leq (\geq) \lambda G(x_1, y_1) + (1-\lambda) G(x_2, y_2)$$

for every $(x_1, y_1), (x_2, y_2) \in A$ and every $\lambda \in [0, 1]$.

Remark 2.2. If G is a C^2 function on the convex set A we have that the function G is convex (concave) if

$$D_d^2 G \geq (\leq) 0 \tag{2.2}$$

on A for every $(d_1, d_2) \in \mathbb{R}^2$, where $D_d^2 G$ is the second order directional derivative in the direction $d = (d_1, d_2)$ of the function G

$$D_d^2 G = d_1^2 G_{xx} + 2 d_1 d_2 G_{xy} + d_2^2 G_{yy}.$$

The condition (2.2) holds if and only if

$$G_{xx} \geq (\leq) 0,\ G_{yy} \geq (\leq) 0,\ G_{xx} G_{yy} - G_{xy}^2 \geq 0.$$

We note

$$\Delta_1 h_j = h_{j+1} - h_j,\ j = 0, ..., n-1,$$
$$\Delta_2 h_j = h_{j+2} - 2h_{j+1} + h_j,\ j = 0, ..., n-2.$$

3. The first family of surfaces

Let $n \in \mathbb{N}$, $n \geq 2$ and $h_i, h \in \mathbb{R}$, $i = 1, ..., n-1$ such that

$$0 = h_n < ... < h_1 < h_0 = h$$

and let $f : [0, 1] \to \mathbb{R}$ be a function with the properties

$$\begin{array}{l} f(0) = h, \\ f(\frac{j}{n}) = h_j,\ j = 1, ..., n-1, \\ f(1) = 0. \end{array} \tag{3.3}$$

From (2.1) and (3.3), we obtain

$$(B_n f)(t) = b_{0n}(t) h + \sum_{j=1}^{n-1} b_{jn}(t) h_j. \tag{3.4}$$

The function in (3.4) has the properties

$$(B_n f)(0) = h,\ (B_n f)(1) = 0.$$

Let $u(x, y)$ be a bivariate positive function such that the curve $C : u(x, y) = 1$ is simple and closed. Let

$$D = \{(x, y) \in \mathbb{R}^2 : 0 \le u(x, y) \le 1\}$$

the domain bounded by the curve C. We assume that D is a convex set with $(0, 0) \in D$ and that the curve $u(x, y) = 0$ is reduced at the point $(0, 0)$.

If we make the substitution

$$t = u(x, y)$$

in (3.4), we obtain the bivariate function

$$F(x, y) = (B_n f)(u(x, y)) = \tag{3.5}$$

$$= b_{0n}(u(x, y))h + \sum_{j=1}^{n-1} b_{jn}(u(x, y))h_j,\ (x, y) \in D.$$

The function F from (3.5) has the properties

$$F|_{\partial D} = 0,\ F(0, 0) = h.$$

It follows that the surfaces $z = F(x, y)$ match the curve $u(x, y) = 1,\ z = 0$ (the surfaces stay on the border of domain D), and the height of the surfaces in the point $(0, 0)$ is h.

Next theorem gives conditions for the monotonicity of the function F.

Theorem 3.3. *If the function u is increasing (decreasing) in the direction (d_1, d_2) then the function F is decreasing (increasing) in the same direction.*

Proof. Taking into account that $\Delta_1 h_j < 0,\ j = 0, ..., n - 1$ and Theorem 2.1 we have

$$F(x + \lambda d_1, y + \lambda d_2) = (B_n f)(u(x + \lambda d_1, y + \lambda d_2)) \le (\ge)(B_n f)(u(x, y)) = F(x, y)$$

for every $(x, y) \in D$ and every $\lambda \in [0, 1]$. Thus F is decreasing (increasing) in the direction (d_1, d_2). □

We give sufficient conditions for the concavity of the function F.

Theorem 3.4. *If $\Delta_2 h_j \le 0,\ j = 0, ..., n - 2$ and the function u is convex then the function F is concave.*

Proof. Using Theorem 2.1 and Theorem 2.2 together with the conditions $\Delta_1 h_j < 0,\ j = 0, ..., n - 1$ and $\Delta_2 h_j \le 0,\ j = 0, ..., n - 2$ we have

$$\begin{aligned} F(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) &= (B_n f)(u(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1 - \lambda)y_2)) \\ &\ge (B_n f)(\lambda u(x_1, y_1) + (1 - \lambda)u(x_2, y_2)) \\ &\ge \lambda(B_n f)(u(x_1, y_1)) + (1 - \lambda)(B_n f)(u(x_2, y_2)) \\ &= \lambda F(x_1, y_1) + (1 - \lambda)F(x_2, y_2)) \end{aligned}$$

for every $(x_1, y_1), (x_2, y_2) \in D$ and every $\lambda \in [0, 1]$.

Thus the function F is concave. □

4. The second family of surfaces

Let $n \in \mathbb{N}, n \geq 2$ and $\widetilde{h}_i, h \in \mathbb{R}$, $i = 1, ..., n-1$ such that

$$0 = \widetilde{h}_0 < \widetilde{h}_1 < ...\widetilde{h}_{n-1} < \widetilde{h}_n = h$$

and let $\widetilde{f} : [0,1] \to \mathbb{R}$ be a function with the properties

$$\begin{array}{l} \widetilde{f}(0) = 0, \\ \widetilde{f}(\frac{j}{n}) = \widetilde{h}_j,\ j = 1, ..., n-1, \\ \widetilde{f}(1) = h. \end{array} \tag{4.6}$$

From (2.1) and the conditions (4.6) we get

$$(B_n\widetilde{f})(t) = \sum_{j=1}^{n-1} b_{jn}(t)\widetilde{h}_j + b_{nn}(t)h. \tag{4.7}$$

The function in (4.7) has the properties

$$(B_n\widetilde{f})(0) = 0,\ (B_n\widetilde{f})(1) = h.$$

Let $\widetilde{u}(x,y)$ be a bivariate positive function such that the curve $\widetilde{C} : u(x,y) = 0$ is simple and closed. Let

$$\widetilde{D} = \{(x,y) \in \mathbb{R}^2 : 0 \leq \widetilde{u}(x,y) \leq 1\}$$

the domain bounded by the curve $\widetilde{C}$. We assume that $\widetilde{D}$ is a convex set with $(0,0) \in \widetilde{D}$ and that the curve $\widetilde{u}(x,y) = 1$ is reduced at the point $(0,0)$.

If we make the substitution

$$t = \widetilde{u}(x,y)$$

in (4.7) we obtain the bivariate function

$$\begin{aligned} \widetilde{F}(x,y) &= (B_n\widetilde{f})(\widetilde{u}(x,y)) \\ &= \sum_{j=1}^{n-1} b_{jn}(\widetilde{u}(x,y))\widetilde{h}_j + b_{nn}(\widetilde{u}(x,y))h,\ (x,y) \in \widetilde{D}. \end{aligned} \tag{4.8}$$

The function $\widetilde{F}$ from (4.8) has the properties

$$\widetilde{F}|_{\partial\widetilde{D}} = 0,\ \widetilde{F}(0,0) = h.$$

It follows that the surfaces $z = \widetilde{F}(x,y)$ match the curve $\widetilde{u}(x,y) = 0,\ z = 0$ (the surfaces stay on the border of domain $\widetilde{D}$), and the height of the surfaces in the point $(0,0)$ is h.

Next theorems give conditions for monotonicity and concavity of the function $\widetilde{F}$.

Theorem 4.5. *If the function $\widetilde{u}$ is increasing (decreasing) in direction (d_1, d_2) then the function $\widetilde{F}$ is increasing (decreasing) in the same direction.*

Theorem 4.6. *If $\Delta_2\widetilde{h}_j \leq 0$, $j = 0, ..., n-2$ and the function $\widetilde{u}$ is concave then the function $\widetilde{F}$ is concave.*

The proofs of Theorem 4.5 and Theorem 4.6 are analogous with the proofs of Theorem 3.3 and Theorem 3.4 respectively.

5. EXAMPLES

Example 5.1. We take

$$u(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

and the domain bounded by a ellipse

$$D=\{(x,y)\in\mathbb{R}^2:\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}.$$

We obtain the surfaces

$$F(x,y)=b_{0n}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)h+\sum_{j=1}^{n-1}b_{jn}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)h_j,\ (x,y)\in D. \tag{5.9}$$

The first order directional derivative of the function u is

$$D_d u=2\left(\frac{d_1x}{a^2}+\frac{d_2y}{b^2}\right).$$

From Theorem 3.3 it follows that if $\frac{d_1x}{a^2}+\frac{d_2y}{b^2}\ge 0(\le 0)$ then the function F is decreasing (increasing) in the direction (d_1,d_2). We have

$$u_{xx}=\frac{2}{a^2}>0,\ u_{yy}=\frac{2}{b^2}>0,\ u_{xy}^2-u_{xx}u_{yy}=-\frac{4}{a^2b^2}<0.$$

Using Remark 2.2 it follows that the function u is convex. From Theorem 3.4 we have that if $\Delta_2 h_j\le 0,\ j=0,...,n-2$ then the surface F is concave.

In Figure 5.a we plot the surface F for $a=3$, $b=2$, $n=3$ and $\{h_0,h_1,h_2,h_3\}=\{4,3,1.7,0\}$.

Example 5.2. We take

$$\widetilde{u}(x,y)=\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}.$$

The function $\widetilde{u}$ generates the same domain as in Example 5.1

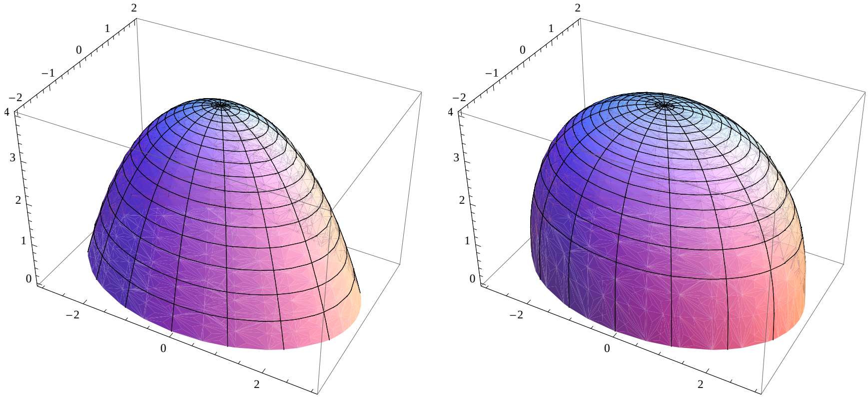
$$\widetilde{D}=\{(x,y)\in\mathbb{R}^2:\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}.$$

We get the surfaces

$$\widetilde{F}(x,y)=b_{nn}\left(\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}\right)h+\sum_{j=1}^{n-1}b_{jn}\left(\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}\right)\widetilde{h}_j,\ (x,y)\in\widetilde{D}. \tag{5.10}$$

The first order directional derivative of the function $\widetilde{u}$ is

$$D_d u=-\frac{1}{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}}\left(\frac{d_1x}{a^2}+\frac{d_2y}{b^2}\right).$$



(a) $z = F(x,y)$ (b) $z = \widetilde{F}(x,y)$

From Theorem 4.5 it follows that if $\dfrac{d_1 x}{a^2} + \dfrac{d_2 y}{b^2} \geq 0 (\leq 0)$ then the function $\widetilde{F}$ is decreasing (increasing) in the direction (d_1, d_2). We have

$$\widetilde{u}_{xx}(x,y) = \frac{y^2 - b^2}{a^2 b^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{3/2}} < 0,\ \widetilde{u}_{yy}(x,y) = \frac{x^2 - a^2}{a^2 b^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{3/2}} < 0,$$

$$\widetilde{u}_{xy}^2 - \widetilde{u}_{xx}\widetilde{u}_{yy} = -\frac{1}{a^2 b^2 \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{2}} < 0.$$

Using the Remark 2.2 it follows that the function $\widetilde{u}$ is concave. From Theorem 4.6 we have that if $\Delta_2 \widetilde{h}_j \leq 0$, $j = 0, ..., n-2$ then the surface $\widetilde{F}$ is concave.

In Figure 5.b we plot the surface $\widetilde{F}$ for $a = 3$, $b = 2$, $n = 3$ and $\{\widetilde{h}_0, \widetilde{h}_1, \widetilde{h}_2, \widetilde{h}_3\} = \{0, 1.7, 3, 4\}$.

References

[1] Coman, Gh., Gânscă, I. and Tâmbulea, L., *Some new roof-surfaces generated by blending interpolation technique*, Studia Univ. Babeş-Bolyai Math. **36** (1991), No. 1, 119–130

[2] Coman, Gh., Gânscă, I. and Tâmbulea, L., *Surfaces generated by blending interpolation*, Studia Univ. Babeş-Bolyai Math. **38** (1993), No. 3, 39–48

[3] Coons, S. A., *Surface for computer aided design of space forms*, Project MAC, Design Div., Dep. of Mech. Engineering, MIT, 1964

[4] Ekeland, I. and Temam, R., *Convex analysis and variational problems*, Classics Appl. Math. **28** (SIAM), Philadelphia, PA, 1999

[5] Floater, M. S., Shape Properties of Tensor-Product Bernstein Polynomials, in *Shape preserving representations in Computer-Aided Geometric Design*, J. M. Pena (ed.), Nova Sci. Publ., New York, 1999, 149–162

[6] Floater, M. S. and Pena, J. M., *Tensor-product Monotonicity Preservation*, Advances in Comp. Math. **9** (1998), 353–362

[7] Lorentz, G. G., *Bernstein polynomials*, Chelsea, New York, 1986

Department of Matematics
Technical University of Cluj Napoca
Memorandumului 28, 400114 Cluj-Napoca, Romania
E-mail address: marius.birou@math.utcluj.ro