CREAT. MATH. INFORM.
21 (2012), No. 1, 41 - 47

Online version at `http://creative-mathematics.ubm.ro/`
Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Ulam-Hyers stability of fixed point equations for singlevalued operators on KST spaces

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ABSTRACT. In this paper we define the notions of Ulam-Hyers stability with respect to a w-distance (in the sense of Kada, Suzuki and Takahashi) and prove several Ulam-Hyers stability results for operators satisfying to a contractive-type condition with respect to w.

1. INTRODUCTION

The existence of fixed points for multivalued operators is being focus of interest for a long time. One of the first results in this sense is Nalder's theorem [13]. For some interesting extensions and generalizations of this result see [5, 11, 12].

On the other hand, in 1976, Caristi [3] proved a fixed point theorem in the framework of complete metric spaces, which is a generalization of the Banach contraction principle. Later, in 1996, O. Kada, T. Suzuki and W. Takahashi [10] introduced the concept of w-distance on a metric space and by using this new concept they obtained a generalization of Caristi's fixed point theorem.

Latter on, T. Suzuki and W. Takahashi, using the setting of a metric space endowed with a w-distance, gave some fixed point results for the so-called multivalued weakly contractive operators (see [25]).

The concept of multivalued weakly Picard operator (briefly MWP operator) was introduced in close connection with the successive approximation method and the data dependence phenomenon for the fixed point set of multivalued operators on complete metric space, by I. A. Rus, A. Petruşel and A. Sântămărian, see [17]. In [15] is presented the theory of multivalued weakly Picard operators in L-spaces.

The Ulam stability of various functional equations have been investigated by many authors (see [1], [2], [6], [7], [8], [9], [14], [18], [21], [22]).

In this paper we define the notions of Ulam-Hyers stability with respect to a w-distance (in the sense of Kada, Suzuki and Takahashi), and prove several Ulam-Hyers stability results for operators satisfying to a contractive-type condition with respect to w.

2. PRELIMINARIES

We denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

For the following notations see I. A. Rus [20] and [21], I. A. Rus, A. Petruşel, A. Sîntămărian [17] and A. Petruşel [15].

Received: 31.10.2010. In revised form: 30.06.2011. Accepted: 30.11.2011

2010 *Mathematics Subject Classification.* 47H10, 54H25, 54C60.

Key words and phrases. *Ulam-Hyers stability, w-distance, fixed point equation, weakly Picard operator, c_w-weakly Picard operator.*

Definition 2.1. Let (X,d) be a metric space and $f : X \to X$ be an operator. By definition, f is weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from $x \in X$ converges, for all $x \in X$ and its limit is a fixed point for f.

If f is WPO, then we consider the operator

$$f^\infty : X \to X \text{ defined by } f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Notice that $f^\infty(X) = Fix(f)$.

Definition 2.2. Let (X, d) be a metric space, $f : X \to X$ be a WPO and $c > 0$ be a real number. By definition, the operator f is c-weakly Picard operator (briefly c-WPO) if and only if $d(x, f^\infty(x)) \le cd(x, f(x))$, for all $x \in X$.

For the theory of weakly Picard operators for the singlevalued case see [20].

In [21] is given the definition of Ulam-Hyers stability as follows.

Definition 2.3. Let (X,d) be a metric space and $f : X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{2.1}$$

is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that: for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \le \varepsilon \tag{2.2}$$

there exists a solution x^* of the equation 2.1 such that

$$d(y^*, x^*) \le c_f \varepsilon.$$

Remark 2.1. If f is a c-weakly Picard operator, then the fixed point equation 2.1 is Ulam-Hyers stable.

The concept of w-distance was introduced by O. Kada, T. Suzuki and W. Takahashi (see [10]) as follows.

Definition 2.4. Let (X, d) be a metric space. Then $w : X \times X \to [0, \infty)$ is called a weak distance (briefly w-distance) on X if the following axioms are satisfied :

(1) $w(x, z) \le w(x, y) + w(y, z)$, for any $x, y, z \in X$;
(2) for any $x \in X$, $w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(3) for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z, x) \le \delta$ and $w(z, y) \le \delta$ implies $d(x, y) \le \varepsilon$.

In order to obtain fixed point results let us recall a crucial result presented in [10].

Lemma 2.1. *Let (X, d) be a metric space and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n), (β_n) be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following hold:*

(1) If $w(x_n, y) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.
(2) If $w(x_n, y_n) \le \alpha_n$ and $w(x_n, z) \le \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.
(3) If $w(x_n, x_m) \le \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.
(4) If $w(y, x_n) \le \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.

By definition, the triple (X, d, w) is a KST-space if X is a nonempty set, $d : X \times X \to \mathbb{R}_+$ is a metric on X and $w : X \times X \to [0, \infty)$ is a w-distance in KST spaces.

Let (X, d, w) be a KST space. We say that (X, d, w) is a complete KST space if the metric space (X, d) is complete.

Some examples of w-distance can be find in [10].

3. Ulam-Hyers stability of fixed point equations for singlevalued operators

We start by give the definition of Ulam-Hyers stability of fixed point equations on KST spaces.

Definition 3.5. Let (X, d, w) be a KST space and $f : X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{3.3}$$

is Ulam-Hyers stable with respect to a w-distance if there exists a real number $c > 0$ such that: for each $\varepsilon > 0$ and each solution y^* of the inequation

$$w(y, f(y)) \leq \varepsilon \tag{3.4}$$

there exists a solution x^* of the equation 3.3 such that

$$w(y^*, x^*) \leq c\varepsilon.$$

Let us denote a c-weakly Picard operator with respect to a w-distance by c_w-weakly Picard operator. Next we define this notion.

Definition 3.6. Let (X, d, w) be a KST space, $f : X \to X$ be a weakly Picard operator and $c > 0$ be a real number. By definition, the operator f is c_w-weakly Picard operator if and only if $w(x, f^\infty(x)) \leq cw(x, f(x))$, for all $x \in X$.

Theorem 3.1. *If f is a c_w-weakly Picard operator, then the fixed point equation* (3.3) *is Ulam-Hyers stable with respect to a w-distance.*

Proof. Let $\varepsilon > 0$ and y^* be a solution of the inequality 3.4. Since f is c_w-weakly Picard operator we have $w(x, f^\infty(x)) \leq cw(x, f(x))$, for every $x \in X$.

If we take $x := y^*$ and $x^* := f^\infty(x)$ we have $w(y^*, x^*) \leq c\varepsilon$. □

Theorem 3.2. *Let (X, d, w) be a complete KST space. Using the previous theorem, results concerning the Ulam-Hyers stability of the fixed point equation* (3.3) *can be given for:*

(1) *Singlevalued weakly r-contraction type operators, i.e., there exists $r \in [0, 1)$ such that $w(f(x), f^2(x)) \leq rw(x, f(x))$, for all $x \in X$, where $c := \dfrac{1}{1-r}$.*

(2) *Singlevalued contraction of weakly Kannan type operators, i.e., there exists $\alpha \in \left[0, \dfrac{1}{2}\right)$ such that $w(f(x), f(y)) \leq \alpha(w(x, f(x)) + w(y, f(y)))$ for all $x, y \in X$, where $c := \dfrac{1-\alpha}{1-2\alpha}$.*

(3) *Singlevalued contraction of weakly Reich type operators, i.e., there exists $a, b, c \in \mathbb{R}_+$ with $a+b+c < 1$ such that $w(f(x), f(y)) \leq aw(x, y) + bw(x, f(x)) + cw(y, f(y))$, for all $x, y \in X$, where $c := \dfrac{1-c}{1-(a+b+c)}$.*

(4) *Singlevalued contraction of weakly Čirič type operators, i.e., there exists $q \in [0, 1)$ such that, for all $x \in X$ we have*

$$w(f(x), f(y)) \leq q\max\{w(x, y), w(x, f(x)), w(y, f(y)), \frac{1}{2}w(x, f(y))\},$$

where $c := \dfrac{1}{1-q}$.

Proof. (1) We must prove that on a KST space a singlevalued weakly r-contraction is a c_w-weakly Picard operator.

Let $x_0 \in X$ be arbitrary chosen. Inductively we construct a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that:

(1) $x_{n+1} = f(x_n)$;
(2) $w(f^n(x_0), f^{n+1}(x_0)) \leq r^n w(x_0, f(x_0))$.

For every $m, n \in \mathbb{N}$ with $m > n$ we obtain the inequality

$$w(f^n(x_0), f^m(x_0)) \leq \frac{r^n}{1-r} w(x_0, f(x_0)).$$

Since (X, d, w) is a complete KST space and using Lemma 2.1 we have that the sequence $\{f^n(x_0)\}$ converge with respect to the metric d to a limit. Let $f^\infty(x_0) = \lim\limits_{n\to\infty} f^n(x_0)$ be the limit of the sequence.

Let $n \in \mathbb{N}$ be fixed. Since $\{f^m(x_0)\}$ converge to the limit $f^\infty(x_0)$ and $w(f(x), \cdot)$ is lower semicontinuous we have

$$w(f^n(x_0), f^\infty(x_0)) \leq \lim_{m\to\infty} \inf w(f^n(x_0), f^m(x_0)) \leq \frac{r^n}{1-r} w(x_0, f(x_0)).$$

Then, by triangle inequality we obtain

$$w(x_0, f^\infty(x_0)) \leq w(x_0, f^n(x_0)) + w(f^n(x_0), f^\infty(x_0)).$$

Then $w(x_0, f^\infty(x_0)) \leq w(x_0, f^n(x_0)) + \dfrac{r^n}{1-r} w(x_0, f(x_0))$.

If we make $n = 1$ we have $w(x_0, f^\infty(x_0)) \leq w(x_0, f(x_0)) + \dfrac{r}{1-r} w(x_0, f(x_0))$. Then $w(x_0, f^\infty(x_0)) \leq \dfrac{1}{1-r} w(x_0, f(x_0))$. Then f is a c_w-weakly Picard operator with $c = \dfrac{1}{1-r}$. Using the Theorem 3.1 follows that the fixed point equation 3.3 is Ulam-Hyers stable with respect to a w-distance.

(2) Next we prove that a singlevalued weakly Kannan type operators is a c_w-weakly Picard operator on a KST space.

Let $x_0 \in X$ be arbitrary chosen. Inductively we construct a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that:

(1) $x_{n+1} = f(x_n)$;
(2) $w(f^n(x_0), f^{n+1}(x_0)) \leq \left(\dfrac{\alpha}{1-\alpha}\right)^n w(x_0, f(x_0))$.

Put $\lambda = \dfrac{\alpha}{1-\alpha}$. Then $0 < \lambda < 1$. For every $m, n \in \mathbb{N}$ with $m > n$ we obtain the inequality

$$w(f^n(x_0), f^m(x_0)) \leq \frac{\lambda^n}{1-\lambda} w(x_0, f(x_0)).$$

Since (X, d, w) is a complete KST space and using Lemma 2.1 we have that the sequence $\{f^n(x_0)\}$ has a limit with respect to the metric d. Let $f^\infty(x_0) = \lim\limits_{n\to\infty} f^n(x_0)$ be the limit of the sequence.

Let $n \in \mathbb{N}$ be fixed. Since $\{f^m(x_0)\}$ converge to the limit $f^\infty(x_0)$ and $w(f(x), \cdot)$ is lower semicontinuous we have

$$w(f^n(x_0), f^\infty(x_0)) \leq \lim_{m\to\infty} \inf w(f^n(x_0), f^m(x_0)) \leq \frac{\lambda^n}{1-\lambda} w(x_0, f(x_0)).$$

Then, by triangle inequality we obtain

$$w(x_0, f^\infty(x_0)) \leq w(x_0, f^n(x_0)) + w(f^n(x_0), f^\infty(x_0)).$$

Then $w(x_0, f^\infty(x_0)) \leq w(x_0, f^n(x_0)) + \dfrac{\lambda^n}{1-\lambda} w(x_0, f(x_0))$.

If we make $n = 1$, we have

$$w(x_0, f^\infty(x_0)) \leq w(x_0, f(x_0)) + \frac{\lambda}{1-\lambda} w(x_0, f(x_0)).$$

Then $w(x_0, f^\infty(x_0)) \leq \dfrac{1}{1-\lambda} w(x_0, f(x_0))$.

If we replace $\lambda = \dfrac{\alpha}{1-\alpha}$ we obtain that $w(x_0, f^\infty(x_0)) \leq \dfrac{1-\alpha}{1-2\alpha} w(x_0, f(x_0))$. Then f is a c_w-weakly Picard operator with $c = \dfrac{1-\alpha}{1-2\alpha}$.

Using the Theorem 3.1 follows that the fixed point equation (3.3) is Ulam-Hyers stable with respect to a w-distance.

(3) Let $x_0 \in X$ be arbitrary chosen. Inductively we construct a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that

(1) $x_{n+1} = f(x_n)$;

(2) $w(f^n(x_0), f^{n+1}(x_0)) \leq \left(\dfrac{a+b}{1-c}\right)^n w(x_0, f(x_0))$.

Put $\beta = \dfrac{a+b}{1-c}$. Then $0 < \beta < 1$. For every $m, n \in \mathbb{N}$ with $m > n$ we obtain the inequality

$$w(f^n(x_0), f^m(x_0)) \leq \frac{\beta^n}{1-\beta} w(x_0, f(x_0)).$$

Since (X, d, w) is a complete KST space and using Lemma 2.1 we have that the sequence $\{f^n(x_0)\}$ has a limit with respect to the metric d. Let $f^\infty(x_0) = \lim\limits_{n\to\infty} f^n(x_0)$ be the limit of the sequence.

Let $n \in \mathbb{N}$ be fixed. Since $\{f^m(x_0)\}$ converge to the limit $f^\infty(x_0)$ and $w(f(x), \cdot)$ is lower semicontinuous we have

$$w(f^n(x_0), f^\infty(x_0)) \leq \lim_{m\to\infty} \inf w(f^n(x_0), f^m(x_0)) \leq \frac{\beta^n}{1-\beta} w(x_0, f(x_0)).$$

Then, by triangle inequality we obtain

$$w(x_0, f^\infty(x_0)) \leq w(x_0, f^n(x_0)) + w(f^n(x_0), f^\infty(x_0)).$$

Then $w(x_0, f^\infty(x_0)) \leq w(x_0, f^n(x_0)) + \dfrac{\beta^n}{1-\beta} w(x_0, f(x_0))$.

If we make $n = 1$ we have

$$w(x_0, f^\infty(x_0)) \leq w(x_0, f(x_0)) + \frac{\beta}{1-\beta} w(x_0, f(x_0)).$$

Then $w(x_0, f^\infty(x_0)) \leq \dfrac{1}{1-\beta} w(x_0, f(x_0))$.

If we replace $\beta = \dfrac{a+b}{1-c}$ we obtain that

$$w(x_0, f^{\infty}(x_0)) \leq \frac{1-c}{1-(a+b+c)} w(x_0, f(x_0)).$$

Then f is a c_w-weakly Picard operator with $c = \dfrac{1-c}{1-(a+b+c)}$.

By Theorem 3.1 we have that the fixed point equation (3.3) is Ulam-Hyers stable with respect to a w-distance.

(4) Let $x_0 \in X$ be arbitrary chosen. Then we have:

(1) $w(f(x_0), f^2(x_0)) \leq qw(x_0, f(x_0))$
(2) $w(f(x_0), f^2(x_0)) \leq qw(x_0, f(x_0))$
(3) $w(f(x_0), f^2(x_0)) \leq qw(f(x_0), f^2(x_0))$
(4) $w(f(x_0), f^2(x_0)) \leq \dfrac{q}{2} w(x_0, f^2(x_0))$

$$w(f(x_0), f^2(x_0)) \leq \frac{q}{2}(w(x_0, f(x_0)) + w(f(x_0)), f^2(x_0))$$

$$w(f(x_0), f^2(x_0)) \leq \frac{q}{2-q}(w(x_0, f(x_0))$$

Then $w(f(x_0), f^2(x_0)) \leq \max\left\{q, \dfrac{q}{2-q}\right\} w(x_0, f(x_0))$. Since $q > \dfrac{q}{2-q}$, for every $q \in [0, 1)$, then $w(f(x_0), f^2(x_0)) \leq qw(x_0, f(x_0))$. On this way, inductively we construct a sequence $(x_n)_{n \in \mathbb{N}} \in X$ such that

(1) $x_{n+1} = f(x_n)$;
(2) $w(f^n(x_0), f^{n+1}(x_0)) \leq q^n w(x_0, f(x_0))$.

For every $m, n \in \mathbb{N}$ with $m > n$ we obtain the inequality

$$w(f^n(x_0), f^m(x_0)) \leq \frac{q^n}{1-q} w(x_0, f(x_0)).$$

Since (X, d, w) is a complete KST space and using Lemma 2.1 we have that the sequence $\{f^n(x_0)\}$ has a limit with respect to the metric d. Let $f^{\infty}(x_0) = \lim\limits_{n \to \infty} f^n(x_0)$ be the limit of the sequence.

Let $n \in \mathbb{N}$ be fixed. Since $\{f^m(x_0)\}$ converge to the limit $f^{\infty}(x_0)$ and $w(f(x), \cdot)$ is lower semicontinuous we have

$$w(f^n(x_0), f^{\infty}(x_0)) \leq \lim_{m \to \infty} \inf w(f^n(x_0), f^m(x_0)) \leq \frac{q^n}{1-q} w(x_0, f(x_0)).$$

Then, by triangle inequality we obtain

$$w(x_0, f^{\infty}(x_0)) \leq w(x_0, f^n(x_0)) + w(f^n(x_0), f^{\infty}(x_0)).$$

Then $w(x_0, f^{\infty}(x_0)) \leq w(x_0, f^n(x_0)) + \dfrac{q^n}{1-q} w(x_0, f(x_0))$.

If we make $n = 1$ we have

$$w(x_0, f^{\infty}(x_0)) \leq w(x_0, f(x_0)) + \frac{q}{1-q} w(x_0, f(x_0)).$$

Then $w(x_0, f^\infty(x_0)) \leq \frac{1}{1-q} w(x_0, f(x_0))$. Then f is a c_w-weakly Picard operator with $c = \frac{1}{1-q}$. Using the Theorem 3.1 follows that the fixed point equation (3.3) is Ulam-Hyers stable with respect to a w-distance. □

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