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On a subclass of analytic functions defined by Ruscheweyh operator

ABDUL RAHMAN SALMAN JUMA and LUMINIŢA-IOANA COTÎRLĂ

ABSTRACT. By using the Ruscheweyh derivative, we have introduced a subclass of analytic functions with negative coefficients in the unit disc. Some properties of analytic function as necessary and sufficient coefficient condition for this class are provided. Distortion bounds, inclusion relation and various properties are also determined.

1. INTRODUCTION

Let $T(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad a_k \geq 0, n \in \mathbb{N} \tag{1.1}$$

which are analytic in the unit disc $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$.

Definition 1.1. A function $f \in T(1)$ is said to be in the subclass $T^*(\alpha)$ of starlike functions of order α if

$$\text{Re}\,\{\frac{zf'(z)}{f(z)}\} > \alpha, \quad z \in \mathbb{U}, \quad 0 \leq \alpha < 1.$$

Definition 1.2. A function $f \in T(1)$ is said to be in the subclass $C(\alpha)$ of convex functions of order α if

$$\text{Re}\,\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha \quad z \in \mathbb{U}, \quad 0 \leq \alpha < 1.$$

Definition 1.3. Let f and g be in $T(n)$. Then we denote by $(f * g)$ the convolution or Hadamard product of f and g that is, if

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad and \quad g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k,$$

then

$$(f * g)(z) = z - \sum_{k=n+1}^{\infty} a_k b_k z^k.$$

Definition 1.4. Let f and g be two analytic functions in $\mathbb{U}$, f is said to be subordinate to g in $\mathbb{U}$ $\quad (f \prec g)$ if there is a function w analytic in $\mathbb{U}$ such that $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$ with $f(z) = g(w(z)), z \in \mathbb{U}$.

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Definition 1.5. The β - th order Ruscheweyh derivative denoted by $D^\beta f$ of function f in $T(n)$ is defined by

$$D^\beta f(z) = \frac{z}{(1-z)^{1+\beta}} * f(z) = z - \sum_{k=n+1}^{\infty} a_k B_k(\beta) z^k,$$

where

$$B_k(\beta) = \frac{(\beta+1)(\beta+2)\cdot \ldots \cdot(\beta+k-1)}{(k-1)!}.$$

On the other hand, a function $f \in T(n)$ is said to be in the class $AJ_n(\beta, \lambda, \mu; A, B)$ if it satisfies the following condition

$$\frac{z(D^\beta f(z))' + \lambda z^2 (D^\beta f(z))''}{(1-\mu)f(z) + \mu z(D^\beta f(z))' + (\lambda-\mu)z^2(D^\beta f(z))''} \prec \frac{1+Az}{1+Bz}, \tag{1.2}$$

$$-1 \le A < B \le 1, 0 \le \mu \le 1, \mu \le \lambda \quad and \quad \beta > -1.$$

The above class contains many classes studied by several authors.
In particular, $AJ_1(0,0,0;-(1-2\alpha),1) \equiv T^*(\alpha)$ and $AJ_1(0,1,1;-(1-2\alpha),1) \equiv C(\alpha)$ were studied by Silverman [5], $AJ_n(0,\lambda,\lambda;-(1-2\alpha),1)$ was studied by Altintas [1], and $AJ_1(0,0,0;A,B)$ and $AJ_1(0,1,1;A,B)$ were studied by Padmanabhan and Ganesan [4].

Our object in the present paper is to investigate characterization theorem, coefficient estimates, a distortion theorem and covering theorem, inclusion theorems and integral operator.

2. Characterization Theorem

We begin by finding the coefficient estimates for the function class $AJ_n(\beta, \lambda, \mu; A, B)$.

Theorem 2.1. *A function $f \in T(n)$ given by* (1.1) *is in the class $AJ_n(\beta, \lambda, \mu; A, B)$ if and only if*

$$\sum_{k=n+1}^{\infty} [(k-1)[k(\mu(1+A) + \lambda(B-A)) + (1-\mu)] - A(1-\mu) + k(B-A\mu)]B_k(\beta)a_k \le B-A,$$

$$-1 \le A < B \le 1, \quad 0 \le \mu \le 1, \quad \mu \le \lambda, \quad \beta > -1. \tag{2.3}$$

The result is sharp for the function f given by

$$f(z) = \tag{2.4}$$

$$z - \frac{B-A}{\{n[(n+1)(\mu(1+A) + \lambda(B-A)) + (1-\mu)] - A(1-\mu) + (n+1)(B-A\mu)\}B_k(\beta)} z^{n+1},$$

$n \in \mathbb{N}$.

Proof. Suppose that $f \in AJ_n(\beta, \lambda, \mu; A, B)$ where f is given by (1.1).
Then by (1.2) we can write

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=n+1}^{\infty}(k-1)(1+\mu(k-1))B_k(\beta)a_k z^k}{(B-A)z - \sum\limits_{k=n+1}^{\infty}\{k(k-1)[B\lambda - A(\lambda-\mu)] + A\mu(1-k) - (A-Bk)\}B_k(\beta)a_k z^k}\right\} < 1, \tag{2.5}$$

or

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=n+1}^{\infty}(k-1)(1+\mu(k-1))B_k(\beta)a_k z^{k-1}}{G(z)}\right\}<1,$$

where

$$G(z)=B-A-\sum_{k=n+1}^{\infty}(k-1)(1+\mu(k-1))B(\beta)a_k z^{k-1}$$

and

$$G(0)=B-A>0.$$

By choosing z on the positive real axis, because $G(0)>0$, $G(z)$ continuous, we have $G(z)>0$, $z\in(0,1)$ and if $z\to 1^-$, z real, (2.5) reduces to the following

$$\frac{\sum\limits_{k=n+1}^{\infty}(k-1)(1+\mu(k-1))B_k(\beta)a_k}{(B-A)-\sum\limits_{k=n+1}^{\infty}\{k(k-1)[B\lambda-A(\lambda-\mu)]+A\mu(1-k)-(A-Bk)\}B_k(\beta)a_k}\leq 1, \tag{2.6}$$

and then (2.3) holds.

Conversely, suppose that (2.3) holds true and let $z\in\partial U=\{z:|z|=1\}$. We have:

$$\left|\frac{\frac{z(D^\beta f(z))'+\lambda z^2(D^\beta f(z))''}{(1-\mu)D^\beta f(z)+\mu z(D^\beta f(z))'+(\lambda-\mu)z^2(D^\beta f(z))''}-1}{B\left(\frac{z(D^\beta f(z))'+\lambda z^2(D^\beta f(z))''}{(1-\mu)D^\beta f(z)+\mu z(D^\beta f(z))'+(\lambda-\mu)z^2(D^\beta f(z))''}\right)-A}\right|$$

$$\leq\frac{\sum\limits_{k=n+1}^{\infty}(k-1)(1+\mu(k-1))B_k(\beta)a_k|z|^k}{(B-A)|z|-\sum\limits_{k=n+1}^{\infty}\{k(k-1)[B\lambda-A(\lambda-\mu)]+A\mu(1-k)-(A-Bk)\}B_k(\beta)a_k|z|^k}$$

$$=\frac{\sum\limits_{k=n+1}^{\infty}(k-1)(1+\mu(k-1))B_k(\beta)a_k}{(B-A)-\sum\limits_{k=n+1}^{\infty}\{k(k-1)[B\lambda-A(\lambda-\mu)]+A\mu(1-k)-(A-Bk)\}B_k(\beta)a_k}\leq 1,$$

$z\in\partial U=\{z:|z|=1\}$.

By Maximum Modulus Theorem, we have $f\in AJ_n(\beta,\lambda,\mu;A,B)$. $\square$

Corollary 2.1. *Let f defined by* (1.1) *in the class $AJ_n(\beta,\lambda,\mu;A,B)$. Then*

$$a_k\leq\frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)},$$

$k=n+1,n+2,\ldots,\quad n\in\mathbb{N}$.

By setting $\lambda=\mu=1$, $\beta=0$, $A=-(1-2\alpha)$, $B=1$ and $n=1$ in Theorem 2.1, we have the following result:

Corollary 2.2. (Silverman [5]) *A function $f\in T(1)$ given by* (1.1) *is in the class $C(\alpha)$ if and only if* $\sum\limits_{k=2}^{\infty}k(k-\alpha)a_k\leq 1-\alpha$.

By setting $\beta=\lambda=\mu=0$, $A=-(1-2\alpha)$, $B=1$ and $n=1$ in Theorem 2.1, we obtain the following result:

Corollary 2.3. (Silverman [5]) *A function $f \in T(1)$ given by (1.1) is in the class $T^*(\alpha)$ if and only if* $\sum_{k=2}^{\infty}(k-\alpha)a_k \le 1-\alpha$.

3. Distortion and covering theorems

Theorem 3.2. *If $f \in AJ_n(\beta,\lambda,\mu;A,B)$, then*

$$r - \frac{(B-A)}{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)}r^{n+1} \le |D^\beta f(z)|$$

$$\le r + \frac{B-A}{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)}r^{n+1}, \qquad (3.7)$$

$|z| = r < 1$.

Proof. Since $f \in AJ_n(\beta,\lambda,\mu;A,B)$, by using Theorem 2.1, we find that

$$\sum_{k=n+1}^{\infty} a_k B_k(\beta)$$

$$\le \frac{B-A}{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)}.$$

Now, from (1.1), we have

$$|D^\beta f(z)| \le |z| + |z|^{n+1}\sum_{k=n+1}^{\infty} a_k B_k(\beta) \le r + r^{n+1}\sum_{k=n+1}^{\infty} a_k B_k(\beta)$$

$$\le r + \frac{B-A}{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)}r^{n+1},$$

and

$$|D^\beta f(z)| \ge |z| - |z|^{n+1}\sum_{k=n+1}^{\infty} a_k B_k(\beta) \ge r - r^{n+1}\sum_{k=n+1}^{\infty} a_k B_k(\beta) \ge$$

$$\ge r - \frac{B-A}{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)}r^{n+1}.$$

□

Theorem 3.3. *If $f \in AJ_n(\beta,\lambda,\mu;A,B)$, then $f \in T^*(\delta)$, where*

$$\delta = 1 - \frac{B-A}{(n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu))B_{n+1}(\beta)}.$$

Proof. We want to show that (2.3) implies that

$$\sum_{k=n+1}^{\infty}(k-\delta)a_k \le 1-\delta,$$

that means

$$\frac{k-\delta}{1-\delta} \le \frac{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]-A(1-\mu)+k(B-A\mu)\}B_k(\beta)}{B-A}, \qquad (3.8)$$

$k \ge n+1$.

This implies that

$$\delta \le 1 - \frac{(k-1)(B-A)}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]-A(1-\mu)+k(B-A\mu)\}B_k(\beta)-(B-A)} = \varphi(k).$$

It is clear that $\varphi(k) \le \varphi(n+1)$, $k \ge n+1$, $n \in \mathbb{N}$, because the degree of the numerator is greater that the degree of the denominator at least one unit and then (3.8) holds true. $\square$

By setting $\beta = 0$, $\lambda = \mu = 1$, $A = -(1-2\alpha)$, $B = 1$ and $n = 1$ in Theorem 3.2, we get the following result:

Corollary 3.4. (Silverman [5]) *If $f \in C(\alpha)$, then $f \in T^*\left(\frac{2}{3-\alpha}\right)$.*
This result is sharp for the extremal function f given by

$$f(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2.$$

4. Extreme points

Theorem 4.4. *Let $f_n(z) = z$ and*

$$f_k(z) = z - \frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)} z^k,$$

$k \ge n+1$, $n \in \mathbb{N}$, $-1 \le A < B < 1$, $0 \le \mu \le 1$, $\mu \le \lambda$, $\beta > -1$.
Then $f \in AJ_n(\beta, \lambda, \mu; A, B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=n+1}^{\infty} \eta_k f_k(z), \tag{4.9}$$

where $\eta_k \ge 0, k \ge n$ and $\sum_{k=n}^{\infty} \eta_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=n}^{\infty} \eta_k f_k(z)$$
$$= \frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)} \eta_k z^k.$$

Therefore,

$$\sum_{k=n+1}^{\infty} \frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)} \eta_k \times$$
$$\times \frac{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)}{B-A}$$
$$= \sum_{k=n+1}^{\infty} \eta_k = 1 - \eta_n \le 1.$$

Then by Theorem 2.1, $f \in AJ_n(\beta, \lambda, \mu; A, B)$.

Conversely, suppose that $f \in AJ_n(\beta, \lambda, \mu; A, B)$. Then

$$a_k \le \frac{B-A}{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)},$$

$k \ge n+1, n \in \mathbb{N}$.

Putting

$$n_k = \frac{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)}{B-A} a_k,$$

$k \geq n+1, n \in \mathbb{N}$ and $\eta_n = 1 - \sum_{k=n+1}^{\infty} \eta_k$, then $f(z) = \sum_{k=n}^{\infty} \eta_k f_k(z)$. □

Corollary 4.5. *The extreme points of the class $AJ_n(\beta, \lambda, \mu; A, B)$ are the functions $f_n(z) = z$ and*

$$f_k(z) = z - \frac{B-A}{\{(k-1)[k(\mu(1+A) + \lambda(B-A)) + (1-\mu)] + k(B-A\mu) - A(1-\mu)\}B_k(\beta)} z^k,$$

$k \geq n+1, n \in \mathbb{N}$.

By setting $\beta = 0$, $\lambda = \mu = 1$, $A = -(1-2\alpha)$, $B = 1$ and $n = 1$ in Theorem 4.1, we get the following result

Corollary 4.6. (Silverman [5]) *The extreme points of the class $C(\alpha)$ are the functions $f_1(z) = z$ and $f_k(z) = z - \frac{1-\alpha}{k(k-\alpha)} z^k$, $k \geq 2$.*

Also by setting $\beta = \lambda = \mu = 0$, $A = -(1-2\alpha)$, $B = 1$ in Corollary 4.1, we get the following result

Corollary 4.7. (Silverman [5]) *The extreme points of the class $T^*(\alpha)$ are the functions $f_1(z) = z$ and $f_k(z) = z - \frac{1-\alpha}{k-\alpha} z^k$, $k \geq 2$.*

Theorem 4.5. *For every $i = 1, ..., m$, let f_i defined by*

$$f_i(z) = z - \sum_{k=n+1}^{\infty} a_{k,i} z^k \quad a_{k,i} \geq 0, \quad i = 1, ..., m, n \in \mathbb{N}$$

be in the class $AJ_n(\beta, \lambda, \mu; A, B)$. Then the function h defined by

$$h(z) = \sum_{i=1}^{m} t_i f_i(z), \quad t_i \geq 0, \quad i = 1, ..., m; \quad \sum_{i=1}^{m} t_i = 1$$

is in the class $AJ_n(\beta, \lambda, \mu; A, B)$.

For the proof of this theorem we apply Theorem 2.1 and we omit its proof.

5. Inclusion relation and integral operator

Theorem 5.6. *Let $0 \leq \mu \leq 1$, $\mu \leq \lambda$, $\beta > -1$, $-1 \leq A < B \leq 1$. Then $AJ_n(\beta, \lambda, \mu; A, B) \subseteq AJ_n(\beta, 0, 0; A_1, B_1)$, where $A_1 \leq 1 - 2m$, $B_1 \geq \frac{A_1 + m}{1 - m}$ and*

$$m = \frac{n(B-A)}{\{n[(n+1)(\mu(1+A) + \lambda(B-A)) + (1-\mu)] + (n+1)(B-A\mu) - A(1-\mu)\}B_{n+1}(\beta)}. \tag{5.10}$$

Proof. Let f defined by (1.1) be in the class $AJ_n(\beta, \lambda, \mu; A, B)$. By making use of the Theorem 2.1, we have

$$\sum_{k=n+1}^{\infty} \frac{\{(k-1)[k(\mu(1+A) + \lambda(B-A)) + (1-\mu)] + k(B-A\mu) - A(1-\mu)\}B_k(\beta)}{B-A} a_k \leq 1. \tag{5.11}$$

Now, we want to find the values A_1, B_1 such that $-1 \le A_1 < B_1 \le 1$ and

$$\sum_{k=n+1}^{\infty} \frac{k(1+B_1)-(1+A_1)}{B_1-A_1} a_k \le 1. \tag{5.12}$$

This holds true if

$$\frac{k(1+B_1)-(1+A_1)}{B_1-A_1}$$

$$\le \frac{\{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]+k(B-A\mu)-A(1-\mu)\}B_k(\beta)}{B-A} = u. \tag{5.13}$$

Simplifying (5.13) we get

$$\frac{B_1-A_1}{1+B_1} \ge \frac{k-1}{u-1}, \quad k \ge n+1, \quad n \in \mathbb{N}. \tag{5.14}$$

It is clear that the right hand side of (5.14) decreases as k increases, therefore it is maximum for $k = n+1$. Thus

$$\frac{B_1-A_1}{1+B_1}$$

$$\ge \frac{n(B-A)}{\{n[(n+1)(\mu(1+A)+\lambda(B-A))+(1-\mu)]+(n+1)(B-A\mu)-A(1-\mu)\}B_{n+1}(\beta)} = m. \tag{5.15}$$

With $m < 1$, fixing A_1 in (5.15), we obtain $B_1 \ge \dfrac{A_1+m}{1-m}$ and $B_1 \le 1$ gives $A_1 \le 1-2m$. □

Theorem 5.7. *Let $0 \le \mu_1 \le 1$, $0 \le \mu_2 \le 1$, $\mu_1 \le \mu_2 \le \lambda_2 \le \lambda_1$, $\beta > -1$, $n \in \mathbb{N}$. Then $AJ_n(\beta, \lambda_1, \mu_1; A, B) \subseteq AJ_n(\beta, \lambda_2, \mu_2; A, B)$.*

The proof of this theorem by making use of Theorem 2.1.

Theorem 5.8. *Let $f \in AJ_n(\beta, \lambda, \mu; A, B)$. Then the Jung-Kim-Srivastava operator*

$$I^\sigma f(z) = z - \sum_{k=n+1}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_k z^k, \quad \sigma > 0$$

is also in the class $AJ_n(\beta, \lambda, \mu; A, B)$.

Proof. Since $\left(\dfrac{2}{n+1}\right)^\sigma \le 1$, $\sigma > 0$, $n \in \mathbb{N}$, we have

$$\sum_{k=n+1}^{\infty} \{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]-A(1-\mu)+k(B-A\mu)\}B_k(\beta)\left(\frac{2}{n+1}\right)^\sigma a_k$$

$$\le \sum_{k=n+1}^{\infty} \{(k-1)[k(\mu(1+A)+\lambda(B-A))+(1-\mu)]-A(1-\mu)+k(B-A\mu)\}B_k(\beta)a_k$$

$$\le B-A,$$

because the coefficient $\left(\dfrac{2}{n+1}\right)^\sigma$ satisfies (2.3).

Therefore, by Theorem 2.1, we have $I^\sigma f \in AJ_n(\beta, \lambda, \mu; A, B)$. □

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DEPARTMENT OF MATHEMATICS
AL-ANBAR UNIVERSITY
RAMADI-IRAQ
E-mail address: dr_juma@hotmail.com

DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF CLUJ NAPOCA
G. BARIŢIU 25, 400027 CLUJ-NAPOCA, ROMANIA
E-mail address: Luminita.Cotirla@yahoo.com
E-mail address: Luminita.Cotirla@math.utcluj.ro