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# The crossing number of $P_5^2 \times P_n$

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ABSTRACT. There are known several exact results concerning crossing numbers of Cartesian products of two graphs. In the paper, we extend these results by giving the crossing number of the Cartesian product $P_5^2 \times P_n$, where $P_n$ is the path of length $n$ and $P_5^2$ is the second power of $P_n$.

## 1. INTRODUCTION

The *crossing number* $cr(G)$ of a simple graph $G$ with vertex set $V$ and edge set $E$ is defined as the minimum number of crossings among all possible drawings of the graph $G$ in the plane. A drawing of $G$ is called *good* if no edge crosses itself, no two edges cross more than once, no two edges incident with the same vertex cross, and no more than two edges cross at a point. It is easy to see that a drawing with minimum number of crossings (an *optimal* drawing) is always a good drawing.

The investigation on crossing numbers of graphs is a classical and however very difficult problem. The exact values of crossing numbers are known only for several graph classes. Most of these results concern Cartesian products of special graphs. (For a definition of Cartesian product $G_1 \times G_2$, see [1].) Let $C_n$ be the cycle on $n$ vertices and $P_m$ be the path on $m+1$ vertices. The crossing numbers were estimated for Cartesian products of paths with all graphs of order at most five [7, 9, 10, 11, 13] as well as for Cartesian products of cycles and all graphs of order at most four [1, 3, 10, 22]. In addition, the crossing numbers of $G \times C_n$ are known for some graphs $G$ on five or six vertices [4, 17, 14, 19, 21]. In [2] Bokal proved the conjecture given by Jendroľ and Ščerbová [7] that $cr(K_{1,n} \times P_m) = (m-1)\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$. The crossing numbers of Cartesian product of stars and all graphs of order three or four are given in [1, 7, 9, 10]. For several graphs of order five, the crossing numbers of Cartesian products with stars are given in [12, 16]. Using the result of Ho [6], it was recently proved in [5] that the crossing number of the Cartesian product of the complete tripartite graph $K_{2,2,2}$ with the star $S_n$ is $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 6n$ for all $n \geq 1$. Except for Cartesian products, very recently the crossing numbers of join of two graphs were studied in [15].

For any positive integer $k$, the *k-power graph* of a graph $G$, denoted by $G^k$, is the graph having the same vertices as $G$, and two vertices of $G^k$ are adjacent if the distance between the corresponding vertices in $G$ is at most $k$. In the paper [20], Patil and Krishnnamurthy established family of graphs for which power graphs have crossing number one. In [23], the crossing numbers of powers of paths were studied. In [18], the crossing numbers of Cartesian products for some second power $P_n^2$ of path $P_n$ with cycles are determined. We start to determine crossing numbers of a new infinite family of graphs, concretely for the Cartesian product $P_5^2 \times P_n$.

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## 2. The graph $P_5^2 \times P_n$

Let $P_5$ be the path of length five. Figure 1(a) shows the power graph $P_5^2$. For the simpler labelling let $F$ denote the graph $P_5^2$, in this paper.

We assume $n \geq 1$ and consider the graph $P_5^2 \times P_n = F \times P_n$ in the following way: it has $6(n+1)$ vertices and edges that are the edges in the $n+1$ copies $F^i$, $i = 0, 1, 2, \ldots, n$, and in the six paths of length $n$. Furthermore, we call the former edges red and the latter ones blue. For $i = 0, 1, \ldots, n$, let $a_i$ and $d_i$ be the vertices of $F^i$ of degree two, $b_i$ and $c_i$ the vertices of degree three, and $p_i$ and $q_i$ the vertices of degree four, as shown in Figure 1. Let $H^i$, $i = 1, 2, \ldots, n$, denote the subgraph of $P_5^2 \times P_n = F \times P_n$ containing the vertices in $F^{i-1}$ and $F^i$ and six edges joining $F^{i-1}$ to $F^i$.

Figure 1(b) shows the drawing of the graph $P_5^2 \times P_n$ with $4(n-1)$ crossings. Hence, we have the upper bound $4(n-1)$ for the crossing number of $P_5^2 \times P_n$.

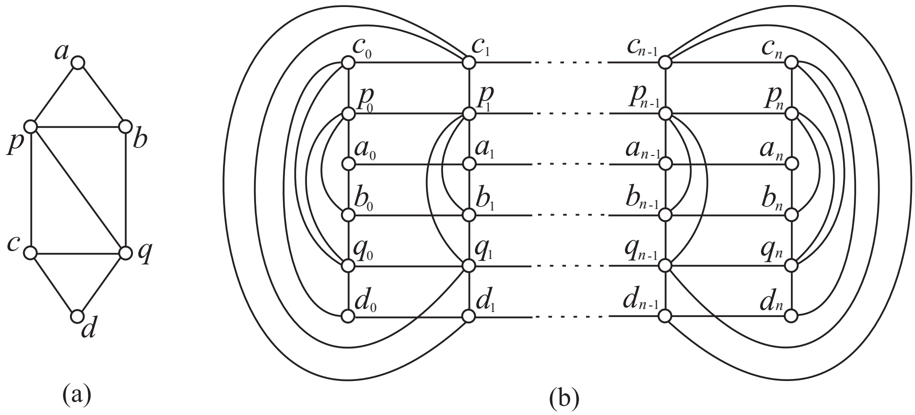


Fig. 1. The graph $P_5^2$, and the Cartesian product $P_5^2 \times P_n$.

For $i = 1, 2, \ldots, n-1$, let $Q^i$ denote the subgraph of $P_5^2 \times P_n$ induced on the vertices of $F^{i-1}$, $F^i$, and $F^{i+1}$. Thus, $Q^i$ has 27 red edges in $F^{i-1}$, $F^i$ and $F^{i+1}$ and 12 blue edges in $H^i$ and $H^{i+1}$. Clearly, $Q^i$ is isomorphic to $P_5^2 \times P_2$. For $x = p, q$, let $S_x$ be the subgraph of $F = P_5^2$ induced on the edges incident with the vertex $x$. The subgraphs $S_p$ and $S_q$ contain one common edge $pq$. So, the graph $F$ consists of $S_p$, $S_q$ and of two additional edges $ab$ and $cd$. In the graph $P_5^2 \times P_n$, let $S_x^i$, $x = p, q$, be the corresponding subgraph of $F^i$, $i \in \{0, 1, \ldots, n\}$, and similarly let $H_x^i$ be the corresponding subgraph of $H^i$, $i \in \{1, 2, \ldots, n\}$. For $x = p, q$, let us denote by $Q_{S_x}^i$ the subgraph of $P_5^2 \times P_n$ induced on the vertices of $S_x^{i-1}$, $S_x^i$, and $S_x^{i+1}$.

## 3. Drawings of $P_5^2 \times P_2$

In this section, we will discuss some properties of drawings of the graph $P_5^2 \times P_n$ and its subgraphs $Q^i$ and $Q_{S_x}^i$. Let $D = D(G)$ be a good drawing of the graph $G$. We denote the number of crossings in $D$ by $cr_D(G)$. For a subgraph $G_i$ of the graph $G$, let $D(G_i)$ be the subdrawing of $G_i$ induced by $D$. For edge-disjoint subgraphs $G_i$ and $G_j$ of $G$, we denote by $cr_D(G_i, G_j)$ the number of crossings of edges of $G_i$ and edges of $G_j$, and by $cr_D(G_i)$ the number of crossings among edges of $G_i$ in $D$. In a good drawing $D$ of the graph $G$, we say that a cycle $C$ *separates* the cycles $C'$ and $C''$ (the vertices of a subgraph $G_i$ not containing vertices of $C$) if $C'$ and $C''$ (the vertices of $G_i$) are contained in different components of $\mathbb{R}^2 \setminus C$.

Assume a good drawing $D$ of the graph $P_5^2 \times P_2 = F \times P_2 = Q^1$ and consider the following types of crossings on the edges of $Q^1$:

(1) a crossing of an edge in $H^1 \cup H^2$ with an edge in $F^1$,

(2) a crossing of an edge in $F^0 \cup H^1$ with an edge in $F^2 \cup H^2$,

(3) a self–intersection in $F^1$,

(4) a crossing of an edge in $F^0$ with an edge in $F^1$,

(5) a crossing of an edge in $F^2$ with an edge in $F^1$.

In the drawing $D$, we define the *force* $f_D(Q^1)$ of $Q^1$ in the following way: every crossing of type (1), (2) or (3) contributes the value 1 to $f_D(Q^1)$ and every crossing of type (4) or (5) contributes the value $\frac{1}{2}$ to $f_D(Q^1)$. Similarly, for $x = p, q$, let $f_D(Q^1_{S_x})$ be the corresponding force of the subdrawing $D(Q^1_{S_x})$ induced from $D(Q^1)$.

In the next lemmas, we establish some properties of drawings of the graph $Q^1 = P_5^2 \times P_2$ as well as for its subgraphs $G^1_{S_x}$.

**Lemma 3.1.** *Let $D = D(Q^1)$ be a good drawing of the graph $Q^1 = P_5^2 \times P_2$ in which do not cross two different subgraphs $F^i$ and $F^j$, $i, j \in \{0, 1, 2\}$. Then $f_D(Q^1_{S_p}) \geq 2$ and $f_D(Q^1_{S_q}) \geq 2$.*

*Proof.* Assume the subgraph $Q^1_{S_p}$ of the graph $Q^1$. By hypothesis, $cr_D(S^0_p, S^1_p) = cr_D(S^0_p, S^2_p) = cr_D(S^1_p, S^2_p) = 0$. The subgraph $Q^1_{S_p}$ contains a subdivision of the complete bipartite graph $K_{3,4}$. As $cr(K_{3,4}) = 2$ [8], in $D(Q^1_{S_p})$ induced by $D$ there are at least two crossings. For $i = 0, 2$, the paths $p_i a_i a_1$, $p_i b_i b_1$, $p_i c_i c_1$ and $p_i q_i q_1$ correspond to the edges incident with a vertex of degree four in $K_{3,4}$. In any good drawing of $K_{3,4}$ do not cross edges incident with a common vertex of degree four. So, a possible crossing among the edges of $S^0_p \cup H^1_p$ joining the vertex $p_0$ with the vertices of $S^1_p$ is not necessary crossing in $D(Q^1_{S_p})$. The same holds for a possible crossing among the edges of $S^2_p \cup H^2_p$ joining the vertex $p_2$ with the vertices of $S^1_p$. Moreover, the edges of $S^1_p$ do not cross each other in $D(Q^1_{S_p})$. Hence, the necessary two crossings in $D(Q^1_{S_p})$ appear between the edges of $H^1_p$ and the edges of $H^2_p$ or between the edges of $S^1_p$ and the edges of $H^1_p \cup H^2_p$. Since every of such crossing contributes the value 1 to the force of $D(Q^1_{S_p})$, $f_D(Q^1_{S_p}) \geq 2$. The similar consideration we can repeat for the subgraph $Q^1_{S_q}$ of $Q^1$, and therefore $f_D(Q^1_{S_q}) \geq 2$, too. □

**Lemma 3.2.** *Let $D = D(Q^1)$ be a good drawing of the graph $Q^1 = P_5^2 \times P_2$ in which every subgraphs $F^i$, $i = 0, 1, 2$, has at most three crossings. Then $f_D(Q^1) \geq 4$.*

*Proof.* Assume that there is a good drawing $D = D(Q^1)$ in which every subgraphs $F^i$, $i = 0, 1, 2$, of $Q^1$ has at most three crossings on its edges and that $f_D(Q^1) < 4$. The following Claim 1 holds for the drawing $D$.

**Claim 1.** $cr_D(F^0, F^1) = cr_D(F^0, F^2) = cr_D(F^1, F^2) = 0$.

*Proof of Claim 1.* Let us assume that $F^0$ and $F^2$ cross each other in $D$. If two 2-connected vertex-disjoint graphs $F^i$ and $F^j$ cross, then they cross at least twice. Thus, $cr_D(F^0, F^2) \geq 2$ and, in the good drawing $D$, $F^0$ separates the vertices of $F^2$ or $F^2$ separates the vertices of $F^0$. Without loss of generality, let $F^2$ separates the vertices of $F^0$. The restriction of at most three crossings on the edges of every $F^i$, $i = 0, 1, 2$, forces that $cr_D(F^0, F^1) = cr_D(F^1, F^2) = 0$. But, in this case, $F^2$ is crossed by at least one edge of $H^1$ joining a vertex of $F^0$ inside $F^2$ with the corresponding vertex of $F^1$. Hence, $cr_D(F^0, F^2) = 2$ and the edges of $F^2$ are crossed three times. These three crossings contribute 3 to $f_D(Q^1)$. It is easy to verify that $cr_D(F^0, F^2) = 2$ only if the subgraph $F^2$ separates a vertex of degree two of $F^0$ from the other five vertices of $F^0$. If the separated vertex of $F^0$ is the vertex $a_0$, then $F^2$ is crossed by all three edges incident with the vertex $a_0$. Consider now the subdrawing $D(Q^1_{S_q})$ of $Q^1_{S_q}$ induced from $D$. None of three considered crossings on the edges of $F^2$ appears in $D(Q^1_{S_q})$. On the other hand, two different subgraphs $S^i_q$ and $S^j_q$, $i,j \in \{0,1,2\}$, do not cross in $D(Q^1_{S_q})$. By Lemma 3.1, $f_D(Q^1_{S_q}) \geq 2$ and, together with three crossings on the edges incident with the vertex $a_0$, we have $f_D(Q^1) \geq 5$. This contradicts the assumption of $f_D(Q^1) < 4$. The same contradiction is obtained when $F^2$ separates the vertex $d_0$ from the other vertices of $F^0$ and therefore, $cr_D(F^0, F^2) = 0$.

Assume now that $F^1$ is crossed in $D$ by $F^0$ or $F^2$. Without loss of generality, let $cr_D(F^0, F^1) \neq 0$. Hence, $cr_D(F^0, F^1) \geq 2$ and $cr_D(F^1, F^2) = cr_D(F^0, F^2) = 0$. We know that if two vertex-disjoint subgraphs $F^0$ and $F^1$ cross each other in $D$, then at least one of them separates the vertices of the other. Consider first that $F^0$ separates the vertices of $F^1$. In this case, at least one edge of $H^2$ joining a separated vertex of $F^1$ with the corresponding vertex of $F^2$ crosses $F^0$, too. So, $cr_D(F^0, F^1) = 2$ and the subgraph $F^0$ separates a vertex of degree two of $F^1$ from the other five vertices of $F^1$. If the separated vertex of $F^1$ is the vertex $a_1$, then $F^0$ is crossed by all three edges of $F^1 \cup H^2$ incident with the vertex $a_1$. These three crossings contribute the value 2 to $f_D(Q^1)$. The same arguments as in the previous paragraph states that in the subdrawing $D(Q^1_{S_q})$ of $Q^1_{S_q}$ induced by $D$, do not cross two different $S^i_q$ and $S^j_q$ and that, using Lemma 3.1, $f_D(Q^1_{S_q}) \geq 2$. Together with the considered three crossings on the edges incident with the vertex $a_1$, we have that $f_D(Q^1) \geq 4$, a contradiction. The same arguments can be used for the case when $F^0$ separates the vertex $d_1$ from the other vertices of $F^1$.

Hence, the only vertices of $F^0$ are separated by $F^1$. This forces that $F^0$ is crossed by only one edge of $F^1$, otherwise $F^0$ separates the vertices of $F^1$. If $cr_D(F^0, F^1) = 2$, then a vertex of degree two of $F^0$ is separated from the other vertices of $F^0$. These two crossings contribute 1 to $f_D(Q^1)$. Assume that $F^1$ separates the vertex $a_0$ from the other five vertices of $F^0$. Let us denote by $Q^1_{-a}$ the subgraph of $Q^1$ obtained by deleting of all edges incident with the vertices $a_0$, $a_1$, and $a_2$. Similarly, let $F^0_{-a}$, $F^1_{-a}$, $F^2_{-a}$, $H^1_{-a}$, and $H^2_{-a}$ be the corresponding subgraphs of $F^0$, $F^1$, $F^2$, $H^1$, and $H^2$, respectively. In the subdrawing $D(Q^1_{-a})$ induced by $D$ the subgraphs $F^0_{-a}$, $F^1_{-a}$, and $F^2_{-a}$ do not cross each other. Lemma 3.1 implies that the subdrawing $D(Q^1_{S_q})$ of $D(Q^1_{-a})$ contributes at least 2 to $f_D(Q^1)$. Hence, $f_D(Q^1) \geq 3$. The edges $b_1p_1$, $p_1c_1$, and $c_1d_1$ of $Q^1_{-a}$ are not included in $D(Q^1_{S_q})$. Thus, they do not cross each other and also they do not cross the edges of $S^1_q$, otherwise this crossing adds at least 1 to $f_D(Q^1)$ and $f_D(Q^1) \geq 4$. This implies that the edges of $F^1_{-a}$ do not cross each other in $D(Q^1_{-a})$. Moreover, as the edges of $F^1$ are crossed twice by the edges incident with the vertex $a_0$ in $D(Q^1)$, at least one of $H^1_{-a}$ and $H^2_{-a}$ cannot cross $F^1_{-a}$ in $D(Q^1_{-a})$. Without loss of generality, let $cr_D(F^1_{-a}, H^1_{-a}) = 0$. In $D(Q^1_{-a})$, the subgraph $F^0_{-a}$ is placed in the region with all five vertices of $F^1_{-a}$ in the view of the subdrawing $D(F^1_{-a})$. This is shown in Figure 2(a), where possible crossings among the edges of $F^0_{-a} \cup H^1_{-a}$ are considered inside the dotted cycle. Hence, in the subdrawing $D(F^0_{-a} \cup H^1_{-a} \cup F^1_{-a})$ there are at most two vertices of $F^1_{-a}$ on the boundary of a region outside $F^1_{-a}$. Inside $F^1_{-a}$ there are at most three vertices of $F^1_{-a}$ on the boundary of every region. Assume now the subdrawing $D(Q^1_{-a})$. The restriction of at the at most three crossings on the edges of $F^1$ forces that $F^2_{-a}$ is placed outside $F^1_{-a}$ in $D(Q^1_{-a})$. As $cr_D(F^2_{-a}, F^0_{-a}) = cr_D(F^2_{-a}, F^1_{-a}) = 0$ the only edges of $H^1_{-a}$ of the subgraph $F^0_{-a} \cup H^1_{-a} \cup F^1_{-a}$ can be crossed by $F^2_{-a}$ in $D(Q^1_{-a})$. If $cr_D(F^2_{-a}, H^1_{-a}) = 0$, then at least three edges of $H^2_{-a}$ cross in the edges of $F^0_{-a} \cup H^1_{-a} \cup F^1_{-a}$. These three crossings, together with two crossings between $F^0$ and $F^1$, enforce $f_D(Q^1) \geq 4$, a contradiction. If $F^2_{-a}$ is placed in two neighbouring regions of $D(F^0_{-a} \cup H^1_{-a} \cup F^1_{-a})$, then $cr_D(F^2_{-a}, H^1_{-a}) \geq 2$ and at least two edges of $H^2_{-a}$ cross in the edges of $F^0_{-a} \cup H^1_{-a} \cup F^1_{-a}$. Hence, $f_D(Q^1) \geq 4$. If $F^2_{-a}$ is placed in more than two neighbouring regions of $D(F^0_{-a} \cup H^1_{-a} \cup F^1_{-a})$, then $cr_D(F^2_{-a}, H^1_{-a}) \geq 4$ and $f_D(Q^1) \geq 4$ again. The same contradiction is obtained also in the case when $F^1$ separates the vertex $d_0$ from the other five vertices of $F^0$.

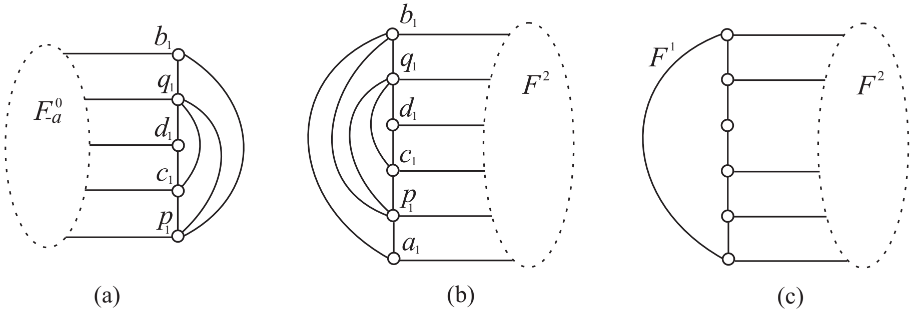


Fig. 2. The subdrawing of $F^0_{-a} \cup H^1_{-a} \cup F^1_{-a}$ (a), the subdrawing of $F^1 \cup H^2 \cup F^2$ (b), and the schematic subdrawing of $F^1 \cup H^2 \cup F^2$ without crossed edge of $H^2$

(c).

Thus, the only possibility for crossings between $F^0$ and $F^1$ is that one edge of $F^1$ is crossed three times by the edges of $F^0$. In this case, $cr_D(F^1) = 0$ and $cr_D(F^1, H^2 \cup F^2) = 0$. The unique such subdrawing of $F^1 \cup H^2 \cup F^2$ is shown in Figure 2(b), where possible crossings among the edges of $H^2 \cup F^2$ are considered inside the dotted cycle. The crossings between $F^0$ and $F^1$ contribute at least $\frac{3}{2}$ to $f_D(Q^1)$. It is easy to verify that if $F^0$ crosses only one edges of $F^1$, then $F^0$ is placed in $D$ in two neighbouring regions of $D(F^1 \cup H^2 \cup F^2)$ with at most three vertices of $F^1$ on the boundary of evry region. So, in $D$, at least three edges of $H^1$ cross the edges of $H^2 \cup F^2$ and every such crossing contributes 1 to $f_D(Q^1)$. Thus, $f_D(Q^1) \geq 4$ again. This proves Claim 1.

By hypothesis, the following Claim 2 holds for the drawing $D$.

**Claim 2.** *The edges of the subgraph $F^1$ cross each other in $D$.*

*Proof of Claim 2.* Assume that $cr_D(F^1) = 0$. The subdrawing $D(F^1)$ induced by $D$ divides the plane into one hexagonal and four triangular regions as shown in Figure 1(a). By Claim 1, $cr_D(F^0, F^1) = cr_D(F^1, F^2) = 0$. If some of $F^0$ and, $F^2$, say $F^0$, is placed in some of triangular regions of the subdrawing $D(F^1)$, then $cr_D(H^1, F^1) \geq 3$ and $F^2$ is placed in $D$ in the hexagonal region of $D(F^1)$ with $cr_D(H^2, F^1) = 0$ as shown in Figure 2(b). It is easy to verify that if the edges of $H^1$ cross $F^1$ only three times, then they cross the edges of $H^2 \cup F^2$, too. This contradicts our assumption that $f_D(Q^1) < 4$.

Thus, both subgraphs $F^0$ and $F^2$ are placed in $D$ in the hexagonal region of the subdrawing $D(F^1)$ and, by the assumption $f_D(Q^1) < 4$, at most one of $H^1$ and $H^2$ crosses $F^1$ more than once. Without loss of generality, let $cr_D(H^2, F^1) \leq 1$. Consider first that $cr_D(H^2, F^1) = 0$. In this case, the edges of $F^2 \cup H^2$ divides the hexagonal region of $D(F^1)$ in such a way that on the boundary of every subregion there are at most two vertices of $F^1$, see Figure 2(b). If $F^0$ is placed in $D$ in some of these regions, then the edges of $H^1$ cross the edges of $F^1 \cup H^2 \cup F^2$ at least four times and $f_D(Q^1) \geq 4$. If $F^0$ is placed in two regions bounded by some edge of $H^2$, then $cr_D(H^2, F^0) \geq 2$ and $H^1$ crosses the edges of $F^1 \cup H^2 \cup F^2$ more than once. Hence, $f_D(Q^1) \geq 4$. If $F^0$ is placed in more than two regions, then $cr_D(H^2, F^0) \geq 4$, which contradicts the assumption of $f_D(Q^1) < 4$ again. Thus, $cr_D(H^2, F^1) = 1$. Consider now the subdrawing of $F^1 \cup H^2 \cup F^2$ without the edge of $H^2$ which crosses $F^1$. The schematic subdrawing of this subgraph without crossed edge of $P^2$ is shown in Figure 2(c). Outside $F^1$ there is only one region with three vertices of $F^1$ on its boundary. The similar analysis as above confirms that, in $D$, any placing of $F^0$ outside $F^1$ forces at least three crossings between $F^0 \cup H^1$ and $F^1 \cup H^2 \cup F^2$. This, together with $cr_D(H^2, F^1) = 1$, implies that $f_D(Q^1) \geq 4$, and the proof of Claim 2 is done.

In the considered drawing $D$ of $Q^1$ with $f_D(Q^1) < 4$, none of $F^0$ and $F^2$ separates the other subgraphs $F^i$ and $F^j$. Otherwise, if $F^0$ separates $F^1$ and $F^2$, then $cr_D(F^0, H^2) \geq 6$, and if $F^2$ separates $F^0$ and $F^1$, then $cr_D(F^2, H^1) \geq 6$. All these crossings are counted in $f_D(Q^1)$. In the rest of the proof we discuss the possible crossings between edges of $H^1$ and edges of $H^2$. We know that none of the subgraphs $F^0$ and $F^2$ separates $F^1$ from the other and, by Claim 1, $F^0$ and $F^2$ do not cross in $D$.

For the case $cr_D(H^1, H^2) = 0$, the unique subdrawing of $F^0 \cup H^1 \cup H^2 \cup F^2$ induced from $D$ is shown in Figure 3(a), where possible crossings among the edges of $F^0 \cup H^1$ are considered in the dotted cycle left and possible crossings among the edges of $F^2 \cup H^2$ are considered in the dotted cycle right. It is easy to verify that, in $D$, the edges of $F^1$ incident with the vertex $p_1$ cross the edges of $H^1 \cup H^2$ at least twice ($cr_D(F^1, F^0 \cup F^2) = 0$) and that the 3-cycle of $F^1$ induced on the vertices $q_1$, $c_1$, and $d_1$ crosses $H^1 \cup H^2$ at least once. These crossings, together with the internal crossing of $F^1$, force $f_D(Q^1) \geq 4$, a contradiction.

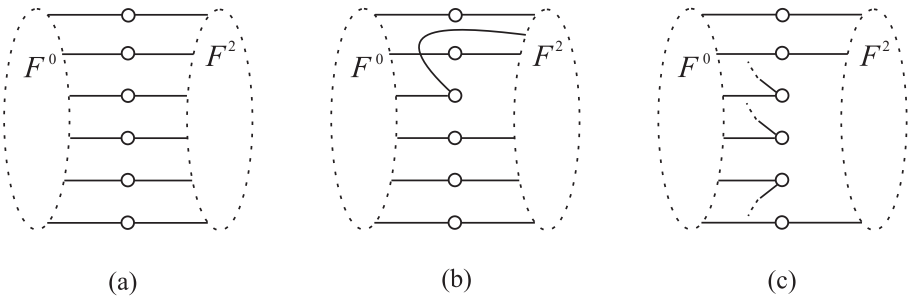


Fig. 1. The possible subdrawings of $F^0 \cup H^1 \cup H^2 \cup F^2$.

Assume now that $cr_D(H^1, H^2) = 1$. In Figure 3(b) there is the unique such drawing with three vertices of $F^1$ on the boundary of one region and with at most two vertices of $F^1$ on the boundaries of the other regions. So, if the 3-cycle induced on the vertices $q_1$, $c_1$, and $d_1$ does not cross $H^1 \cup H^2$ in $D$, then it is placed in the region of $D(F^0 \cup H^1 \cup H^2 \cup F^2)$ with three vertices of $F^1$ on its boundary. But, in this case, the vertex $p_1$ is not on the boundary of the region with three vertices of $F^1$ and $S^1_p$ crosses the edges of $H^1 \cup H^2$ at least twice. Hence, $f_D(Q^1) \geq 4$. If the 3-cycle induced on the vertices $q_1$, $c_1$, and $d_1$ crosses $H^1 \cup H^2$ in $D$, then also $cr_D(S^1_p, H^1 \cup H^2) \neq 0$ and $f_D(Q^1) \geq 4$ again.

The last possibility is that $cr_D(H^1, H^2) \geq 2$. Thus, on the edges of $H^1 \cup H^2$ there are at least two crossings counted in $f_D(Q^1)$. The assumption of $f_D(Q^1) < 4$ forces that $cr_D(S^1_p, H^1 \cup H^2) = 0$, and this is possible only if in $D(F^0 \cup H^1 \cup H^2 \cup F^2)$ there is a region with five vertices of $F^1$ on its boundary. The drawing in Figure 3(c) shows that, in such a case, $cr_D(F^0 \cup H^1, F^2 \cup H^2) \geq 3$ and therefore, $f_D(Q^1) \geq 4$. This proves that, by the assumption of Lemma 3.2, there is no good drawing of $Q^1$ with $f_D(Q^1) < 4$, and the proof is done. $\square$

## 4. The crossing number of $P_5^2 \times P_n$

It is easy to verify that the graph $P_5^2 \times P_1$ is planar. The aim of this section is to give the crossing number of the graph $P_5^2 \times P_n$. First we establish the crossing number of the graph $P_5^2 \times P_2$.

**Lemma 4.3.** $cr(P_5^2 \times P_2) = 4$.

*Proof.* If there is a good drawing $D = D(P_5^2 \times P_2)$ with less than three crossings, then every subgraph $F^i$, $i = 0, 1, 2$, is crossed at most three times. Thus, by Lemma 3.2, $f_D(Q^1) \geq 4$. As the number of crossings in $D$ is not less than the force of $D$, the proof is done. □

The next Lemma 4.4 is necessary for proving the main result of this paper.

**Lemma 4.4.** *If $D$ is a good drawing of $P_5^2 \times P_n$, $n \geq 2$, in which every copy of $P_5^2$ has at most three crossings on its edges, then $D$ has at least $4(n-1)$ crossings.*

*Proof.* In the drawing $D$ of the graph $P_5^2 \times P_n$, we define the *total force* of the drawing as the sum of $f_D(Q^i)$ of all subdrawings $D(Q^i)$, $i = 1, 2, \ldots, n-1$, induced by $D$. It is readily seen that, in $D$, every crossing of types (1), (2), and (3) appears only on the edges of the subgraph $Q^i$. For $i = 2, 3, \ldots, n-1$, every crossing of type (4) in $Q^i$ appears and also in $Q^{i-1}$ as a crossing of type (5). Similarly, for $i = 1, 2, \ldots, n-2$, every crossing of type (4) in $Q^i$ appears and also in $Q^{i+1}$ as a crossing of type (4). As every crossing of type (1), (2) or (3) contributes the value 1 to $f_D(Q^i)$ and every crossing of type (4) or (5) contributes the value $\frac{1}{2}$ to $f_D(Q^i)$, it is easy to see that the number of crossings in the drawing $D$ is not less then the total force of the drawing. By Lemma 3.2, the total force of the drawing $D$ is at least $4(n-1)$. Hence, in $D$ there are at least $4(n-1)$ crossings. □

**Theorem 4.1.** *For $n \geq 2$, $cr(P_5^2 \times P_n) = 4(n-1)$.*

*Proof.* The drawing in Figure 1(b) shows that $cr(P_5^2 \times P_n) \leq 4(n-1)$ for $n \geq 2$. We prove the reverse inequality by induction on $n$. By Lemma 4.3, $cr(P_5^2 \times P_2) = 4$, so the result is true for $n = 2$. Assume that it is true for $n = k$, $k \geq 2$, and suppose that there is a drawing of $P_5^2 \times P_{k+1}$ with fewer than $4((k+1)-1)) = 4k$ crossings. By Lemma 4.4, some $F^i$ must then be crossed at least three times. By the removal of all edges of this $F^i$, we obtain a subdivision of $P_5^2 \times P_k$ with fewer than $4(k-1)$ crossings or the subgraph isomorphic to $P_5^2 \times P_k$ with fewer than $4(k-1)$ crossings. This contradicts the induction hypothesis. □

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