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Ulam-Hyers stability for partial differential equations

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ABSTRACT. Using the weakly Picard operator technique, we will present some Ulam-Hyers stability results for some partial differential equations.

1. INTRODUCTION

The Ulam-Hyers stability (Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin,...) of various functional equations have been investigated by many authors (see [1], [3], [5], [6], [9]-[11], [18], [22], [23]). There are some results for differential equation ([12], [14], [15], [17], [29]), integral equation ([13],[28]), and for difference equation ([2], [20], [21]).

The aim of this paper is to present Ulam-Hyers stability results for some problems associated to partial differential equations.

2. ULAM-HYERS STABILITY VIA WEAKLY PICARD OPERATORS

We will present first some notions and results from the weakly Picard operator theory (see [26]; see also [32], pp. 119-126).

Let (X, d) be a metric space and $f : X \to X$ be an operator. We denote by $F_f := \{x \in X \mid f(x) = x\}$, the fixed point set of the operator f. By definition, f is a weakly Picard operator if the sequence of successive approximations, $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

If f is a weakly Picard operator, then we consider the operator $f^\infty : X \to X$ defined by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$. It is obvious that $f^\infty(X) = F_f$. Moreover, f^∞ is a set retraction of X to F_f.

If f is a weakly Picard operator and $F_f = \{x^*\}$, then by definition f is a Picard operator. In this case f^∞ is the constant operator, $f^\infty(x) = x^*$, for all $x \in X$. The following class of weakly Picard operators is very important in our considerations.

Definition 2.1. Let (X, d) be a metric space and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function, continuous in 0 and with $\psi(0) = 0$. An operator $f : X \to X$ is said to be a ψ-weakly Picard operator if it is weakly Picard and

$$d(x, f^\infty(x)) \le \psi(d(x, f(x))), \text{ for all } x \in X.$$

In the case that $\psi(t) = ct$, for $t \in \mathbb{R}$ (for some $c > 0$), then we say that f is a c-weakly Picard operator.

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Example 2.1. Let (X,d) be a complete metric space and $f : X \to X$ an operator with closed graphic. We suppose that f is graphic α-contraction, i.e.,

$$d(f^2(x), f(x)) \leq \alpha d(x, f(x)), \text{ for all } x \in X.$$

Then f is a c-weakly Picard operator, with $c = (1-\alpha)^{-1}$.

Example 2.2. Let (X,d) be a complete metric space, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a function and $f : X \to X$ an operator with closed graphic. We suppose that:

(i) f is a φ-Caristi operator, i.e., $d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),$ for all $x \in X$;

(ii) there exists $c > 0$ such that $\varphi(x) \leq cd(x, f(x)),$ for all $x \in X$.

Then f is a c-weakly Picard operator.

On the other hand, by the analogy with the notion of the Ulam-Hyers stability in the theory of functional equation (see [10], [11], [3], [1], [5], [6]-[9], [18], [22], [23],...) we consider the following concept.

Definition 2.2. Let (X,d) be a metric space and $f : X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{2.1}$$

is generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{2.2}$$

there exists a solution x^* of the equation (2.1) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If, in particular, there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, the equation (2.1) is said to be Ulam-Hyers stable.

The following results are important for our further considerations.

Lemma 2.1 (I. A. Rus [30])**.** *If f is a ψ-weakly Picard operator, then the fixed point equation (2.1) is generalized Ulam-Hyers stable.*

Lemma 2.2 (I. A. Rus [30])**.** *Let (X,d) be a metric space, $f : X \to X$ be operator and $X = \bigcup_{i \in I} X_i$ a partition of X such that $f(X_i) \subset X_i$, $\forall\, i \in I$. If the equation (2.1) is Ulam-Hyers stable in each (X_i, d), $i \in I$, then it is Ulam-Hyers stable in (X,d).*

3. Ulam-Hyers stability for partial differential equations

We will consider first the Dirichlet problem associated to a nonlinear elliptic equation.

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth border $\partial\Omega$. Consider the following problem:

$$\Delta u = f(x, u(x)) \tag{3.3}$$

$$u_{|\partial\Omega} = 0, \tag{3.4}$$

where f is a continuous function on $\overline{\Omega} \times \mathbb{R}$.

Throught this section we will denote by $||\cdot||_C$ the supremum norm in $C(\overline{\Omega}, \mathbb{R})$.

The following auxiliary result is well-known in the theory of partial differential equations, see for example I. A. Rus [25], page 212.

Lemma 3.3. *In the above conditions, the Dirichlet problem* $(3.3) + (3.4)$ *is equivalent to the following integral equation*

$$u(x) = -\int_{\Omega} G(x,s) f(s,u(s)) ds \tag{3.5}$$

where G denotes the usual Green function correspoonding to the Laplace operator.

We recall now some known notions and a fixed point result.

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function (see [31]) if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is continuous in 0.

Theorem 3.1 (J. Matkowski [16], I. A. Rus [31]). *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a* φ*-contraction. Then* $F_f = \{x^*\}$ *and* $f^n(x_0) \to x^*$ *when* $n \to \infty$, *for all* $x_0 \in X$, *i.e., f is a Picard operator.*

Notice that, if in the above result, we additionally suppose that the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and onto, then the operator f is ψ-weakly Picard. Indeed, we have $d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + \varphi(d(x, x^*))$. Hence

$$d(x, x^*) \leq \psi^{-1}(d(x, f(x)), \text{ for each } x \in X.$$

Our first main result is the following existence, uniqueness and stability result for the Dirichlet problem $(3.3) + (3.4)$.

Theorem 3.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *such that its border* $\partial\Omega$ *is sufficiently smooth. Denote by G denotes the usual Green function correspoonding to the Laplace operator. Suppose that:*

(a) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$;

(b) there exist $p \in C(\overline{\Omega}, \mathbb{R}_+)$ *with* $\sup\limits_{x \in \overline{\Omega}} \displaystyle\int_{\Omega} G(x,s)p(s)ds \leq 1$ *and a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, *such that for each* $s \in \overline{\Omega}$ *and each* $u, v \in \mathbb{R}$ *we have that*

$$|f(s,u) - f(s,v)| \leq p(s)\varphi(|u - v|).$$

Then, the Dirichlet problem $(3.3) + (3.4)$ *has a unique solution* $u^* \in C(\overline{\Omega}, \mathbb{R})$. *Moreover, if the function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ *is strictly increasing and onto, then the Dirichlet problem* $(3.3) + (3.4)$ *is generalized Ulam-Hyers stable with function* ψ^{-1}, *i.e., for each* $\varepsilon > 0$ *and for each* ε*-solution* y^* *of the Dirichlet problem* $(3.3) + (3.4)$ *we have that*

$$|u^*(x) - y^*(x)| \leq \psi^{-1}(\varepsilon), \textit{ for each } x \in \overline{\Omega}.$$

Proof. Consider the operator $A : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R})$ given by

$$Au(x) := -\int_{\Omega} G(x,s) f(s,u(s)) ds.$$

Using this notation, by Lemma 3.3, the Dirichlet problem $(3.3) + (3.4)$ is equivalent with the fixed point equation $u = Au$. Next we have:

$|Au(x) - Av(x)| \leq \int_{\Omega} |G(x,s)||f(s,u(s)) - f(s,v(s))| ds \leq \int_{\Omega} |G(x,s)| p(s)\varphi(|u(s) - v(s)|) ds \leq \varphi(||u - v||_C) \int_{\Omega} G(x,s)p(s)ds \leq \varphi(||u - v||_C)$.

Hence, by taking the supremum over $x \in \overline{\Omega}$ we get that

$$||Au - Av||_C \leq \varphi(||u - v||_C), \text{ for all } u, v \in C(\overline{\Omega}, \mathbb{R}).$$

Now the first conclusion follows from Theorem 3.1.

For the second conclusion, let $\varepsilon > 0$ and let $y^* \in C(\overline{\Omega}, \mathbb{R})$ be an ε-solution for $(3.3) + (3.4)$, i.e.,

$$|y^*(x) + \int_\Omega G(x,s) f(s, y^*(s)) ds| \leq \varepsilon, \text{ for each } x \in \overline{\Omega}.$$

Since A is a ψ-weakly Picard operator, the second conclusion follows from Lemma 2.1, i.e.,

$$|y^*(x) - u^*(x)| \leq \psi^{-1}(\varepsilon), \text{ for each } x \in \overline{\Omega}.$$

□

Remark 3.1. Theorem 3.2 generalizes some known results in the literature, see Theorem 16.2.1 in I. A. Rus [25].

A second result concerns with the case of a Dirichlet problem for a partial differential equation with modified argument.

Consider now the following problem:

$$\Delta u = f(x, u(g(x))) \tag{3.6}$$

$$u_{|\partial\Omega} = 0 \tag{3.7}$$

where f is a continuous function on $\overline{\Omega} \times \mathbb{R}$ and $g \in C(\overline{\Omega}, \overline{\Omega})$.

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ such that its border $\partial\Omega$ is sufficiently smooth. Denote by G denotes the usual Green function corresponding to the Laplace operator. Suppose that:*

(a) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $g \in C(\overline{\Omega}, \overline{\Omega})$;

(b) there exist $p \in C(\overline{\Omega}, \mathbb{R}_+)$ with $\sup_{x \in \overline{\Omega}} \int_\Omega G(x,s) p(s) ds \leq 1$ and a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each $s \in \overline{\Omega}$ and each $u, v \in \mathbb{R}$, we have that

$$|f(s,u) - f(s,v)| \leq p(s)\varphi(|u - v|);$$

Then, the Dirichlet problem $(3.6) + (3.7)$ has a unique solution $u^* \in C(\overline{\Omega}, \mathbb{R})$. Moreover, if the function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) := t - \varphi(t)$ is strictly increasing and onto, then the Dirichlet problem $(3.6) + (3.7)$ is generalized Ulam-Hyers stable with function ψ^{-1}, i.e., for each $\varepsilon > 0$ and for each ε-solution y^* of the Dirichlet problem $(3.6) + (3.7)$ we have that

$$|u^*(x) - y^*(x)| \leq \psi^{-1}(\varepsilon), \textit{ for each } x \in \overline{\Omega}.$$

Proof. Consider the operator $A : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R})$ given by

$$Au(x) := -\int_\Omega G(x,s) f(s, u(g(s))) ds.$$

Then, the Dirichlet problem $(3.6) + (3.7)$ is equivalent with the fixed point equation $u = Au$.

Next we have:

$$|Au(x) - Av(x)| \leq \int_\Omega |G(x,s)||f(s, u(g(s))) - f(s, v(g(s)))| ds \leq$$

$$\int_\Omega |G(x,s)| p(s) \varphi(|u(g(s)) - v(g(s))|) ds \leq \varphi(||u - v||_C) \int_\Omega G(x,s) p(s) ds \leq \varphi(||u - v||_C).$$

Hence, by taking the supremum over $x \in \overline{\Omega}$ we get that

$$||Au - Av||_C \leq \varphi(||u - v||_C), \text{ for all } u, v \in C(\overline{\Omega}, \mathbb{R}).$$

Now the first conclusion follows from Theorem 3.1.

For the second conclusion, let $\varepsilon > 0$ and let $y^* \in C(\overline{\Omega}, \mathbb{R})$ be an ε-solution for $(3.6) + (3.7)$, i.e.,

$$|y^*(x) + \int_\Omega G(x,s)f(s,y^*(g(s)))ds| \leq \varepsilon, \text{ for each } x \in \overline{\Omega}.$$

Since A is a ψ-weakly Picard operator, the second conclusion follows from Lemma 2.1, i.e.,

$$|y^*(x) - u^*(x)| \leq \psi^{-1}(\varepsilon), \text{ for each } x \in \overline{\Omega}.$$

□

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