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# The generalization of certain results for Szász-Mirakjan-Schurer operators

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ABSTRACT. The present article continues earlier research by authors, in order to reach two goals. Firstly, we give a general formula concerning calculation of the test functions by Szász-Mirakjan-Schurer operators and secondly, we establish a Voronovskaja type theorem, the uniform convergence and the order of approximation using the modulus of continuity for the same operators.

## 1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
For any $n \in \mathbb{N}$ define the operators $S_n : C_2[0, +\infty) \to C[0, +\infty)$ given by

$$S_n(f; x) = \sum_{k=0}^{\infty} s_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

for any $x \in [0, +\infty)$, where $s_{n,k}(x) = e^{-nx} \dfrac{(nx)^k}{k!}$ and

$$C_2[0, +\infty) = \left\{ f \in C[0, +\infty) : \lim_{x \to \infty} \frac{f(x)}{1 + x^2} \text{ exists and is finite } \right\}$$

is the space endowed with the norm $\|f\|_2 := \sup\limits_{x \geq 0} \dfrac{|f(x)|}{1 + x^2}$.
The operators $(S_n)_{n \in \mathbb{N}}$ are called Mirakjan-Favard-Szász operators and were first introduced in 1941 by G. M. Mirakjan [7]. They were intensively studied in 1944 by J. Favard [3] and in 1950 by O. Szász [12].
In the following let $p \in \mathbb{N}_0$ and the operators $S_{n,p} : C_2[0, +\infty) \to C[0, +\infty)$ be defined by

$$S_{n,p}(f; x) = \sum_{k=0}^{\infty} s_{n,p,k}(x) f\left(\frac{k}{n}\right), \tag{1.2}$$

for any $x \in [0, +\infty)$ and any $n \in \mathbb{N}$, where

$$s_{n,p,k}(x) = e^{-(n+p)x} \frac{((n+p)x)^k}{k!} = s_{n+p,k}(x).$$

The operators $(S_{n,p})_{n \in \mathbb{N}}$ are called Szász-Mirakjan-Schurer operators and were first introduced in 1962, by F. Schurer [9], see also F. Schurer [10] and P. C. Sikkema [11]. Note that

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the series on the right hand side of (1.2) is absolutely convergent for any $f \in C_2[0,+\infty)$, i.e., the following

$$\begin{aligned} e^{-(n+p)x}\sum_{k=0}^{\infty}\frac{((n+p)x)^k}{k!}\left|f\left(\frac{k}{n}\right)\right| &\le \|f\|_2\cdot e^{-(n+p)x}\sum_{k=0}^{\infty}\frac{((n+p)x)^k}{k!}\left(1+\frac{k^2}{n^2}\right)\\ &= \|f\|_2\cdot e^{-(n+p)x}\left(\sum_{k=0}^{\infty}\frac{((n+p)x)^k}{k!}+\sum_{k=0}^{\infty}\frac{((n+p)x)^k}{k!}\cdot\frac{k(k-1)+k}{n^2}\right)\\ &= \|f\|_2\cdot e^{-(n+p)x}\left(e^{(n+p)x}+\frac{(n+p)^2x^2}{n^2}\cdot e^{(n+p)x}+\frac{(n+p)x}{n^2}\cdot e^{(n+p)x}\right)\\ &= \|f\|_2\cdot\left(1+\frac{(n+p)x}{n^2}+\frac{(n+p)^2x^2}{n^2}\right) \end{aligned}$$

hold.

**Remark 1.1.** More results and properties concerning Szász-Mirakjan-Schurer operators can be found in monographs [1] and [2].

The purpose of this paper is to generalize certain results for Szász-Mirakjan-Schurer operators. Firstly, we give a general formula concerning calculation of the test functions and next, taking this into account we establish the moments up to the fourth order. Secondly, about some known results which we will cite at the adequate moment, we shall prove uniform convergence, general Voronovskaja type formulas and the order of approximation up to twice continuously differentiable functions, for the same operators.

## 2. Preliminaries

Of the greatest utility in the calculus of finite differences, in number theory, in the summation of series are the numbers introduced in 1730 by J. Stirling in his *Methodus differentialis* [13], subsequently called "Stirling numbers" of the first and second kind. For any $x \in \mathbb{R}$ and any $n \in \mathbb{N}_0$, let $(x)_n := \prod_{i=0}^{n-1}(x-i)$, where $(x)_0 := 1$ be the falling factorial denoted by Pochhammer symbol. It is well known that

$$x^j = \sum_{i=0}^{j} S(j,i)(x)_i \tag{2.3}$$

holds, for any $x \in \mathbb{R}$ and any $j \in \mathbb{N}_0$, where $S(j,i)$ are the Stirling numbers of second kind. Now, let $i,j \in \mathbb{N}_0$ be natural numbers, then the Stirling numbers of second kind have the following properties:

$$S(j,i) := \begin{cases} 1, & \text{if } j=i=0;\ j=i \text{ or } j>1, i=1\\ 0, & \text{if } j>0, i=0\\ 0, & \text{if } j<i\\ i\cdot S(j-1,i)+S(j-1,i-1), & \text{if } j,i>1. \end{cases} \tag{2.4}$$

Let $e_j(x) = x^j$, with $j \in \mathbb{N}_0$ be the test functions. A further way to derive information on all moments of linear operators is given in:

**Proposition 2.1.** [4] *For a linear operator $L$ and $j \in \mathbb{N}_0$ one has*

$$L\left((e_1-x)^j;x\right) = L(e_j;x) - \sum_{i=0}^{j-1} \binom{j}{i} x^{j-i} L\left((e_1-x)^i;x\right).$$

We recall some results from [8], which we shall use afterwards in the paper. Let $I, J$ be real intervals and $I \cap J \neq \emptyset$. For any $n, k \in \mathbb{N}_0$, $n \neq 0$ consider the functions $\varphi_{n,k} : J \to \mathbb{R}$, with the property that $\varphi_{n,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{n,k} : E(I) \to \mathbb{R}$. For any $n \in \mathbb{N}$ define the operator $L_n : E(I) \to F(J)$, by

$$L_n(f;x) = \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}(f), \tag{2.5}$$

where $E(I)$ is a linear space of real-valued functions defined on $I$, for which the operators (2.5) are convergent and $F(J)$ is a subset of the set of real-valued functions defined on $J$.

**Remark 2.2.** [8] The operators $(L_n)_{n\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For $n \in \mathbb{N}$ and $i \in \mathbb{N}_0$ define $T^*_{n,i}$ by

$$T^*_{n,i}(L_n;x) = n^i L_n\left(\psi_x^i;x\right) = n^i \sum_{k=0}^{\infty} \varphi_{n,k}(x) A_{n,k}\left(\psi_x^i\right), \quad x \in I \cap J. \tag{2.6}$$

In what follows $s \in \mathbb{N}_0$ is even and we assume that the next two conditions:

- there exists the smallest $\alpha_s,\ \alpha_{s+2} \in [0, +\infty)$, so that

$$\lim_{n\to\infty} \frac{T^*_{n,j}(L_n;x)}{n^{\alpha_j}} = B_j(x) \in \mathbb{R}, \tag{2.7}$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \tag{2.8}$$

- $I \cap J$ is an interval

hold.

**Theorem 2.1.** [8] *If $f \in E(I)$ is a function $s$ times differentiable in a neighborhood of $x \in I \cap J$, then*

$$\lim_{n\to\infty} n^{s-\alpha_s}\left(L_n(f;x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i! \cdot n^i} T^*_{n,i}(L_n;x)\right) = 0. \tag{2.9}$$

*Assume that $f$ is a $s$ times differentiable function on $I$ and there exists an interval $K \subset I \cap J$, such that, there exist $n(s) \in \mathbb{N}$ and the constants $k_j \in \mathbb{R}$ depending on $K$, so that for $n \geq n(s)$ and $x \in K$, the following*

$$\frac{T^*_{n,j}(L_n;x)}{n^{\alpha_j}} \leq k_j, \tag{2.10}$$

*hold, for $j \in \{s, s+2\}$. Then, the convergence expressed by* (2.9) *is uniform on $K$ and moreover*

$$n^{s-\alpha_s}\left|L_n(f;x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{i! \cdot n^i} T^*_{n,i}(L_n;x)\right| \leq \frac{1}{s!}(k_s + k_{s+2}) \cdot \omega_1\left(f^{(s)}, \frac{1}{\sqrt{n^{2+\alpha_s-\alpha_{s+2}}}}\right), \tag{2.11}$$

*for any $x \in K$ and $n \geq n(s)$, where $\omega_1(f, \cdot)$ denotes the modulus of continuity* [1], *of the function $f$.*

## 3. MAIN RESULTS

In order to reach the first aim of this paper, we need to recall the following result, which had been proved earlier by S. Karlin and Z. Ziegler [5]. The authors established a general formula concerning the computation of the test functions by Bernstein operators, in terms of the Stirling numbers of second kind

$$B_n(e_j;x)=\frac{1}{n^j}\sum_{i=1}^{j}S(j,i)(n)_i x^i,$$

for any $j,n\in\mathbb{N}$, $j\leq n$. Having in mind this result, we want to establish an appropriate relation for Szász-Mirakjan-Schurer operators, given by the following:

**Proposition 3.2.** *For any $j,n\in\mathbb{N}$ and any $x\in[0,+\infty)$, the following holds*

$$S_{n,p}(e_j;x)=\frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)((n+p)x)^{j-i}. \tag{3.12}$$

*Proof.* For the proof of proposition we use the relation (2.3) expressed in the following form

$$x^j=\sum_{i=0}^{j-1}S(j,j-i)(x)_{j-i}, \tag{3.13}$$

because $S(j,0)=0$, see (2.4). Taking into account the relation (3.13), we get

$$\begin{aligned}
S_{n,p}(e_j;x)&=e^{-(n+p)x}\sum_{k=0}^{\infty}\frac{((n+p)x)^k}{k!}\cdot\frac{k^j}{n^j}\\
&=\frac{e^{-(n+p)x}}{n^j}\sum_{k=0}^{\infty}\frac{((n+p)x)^k}{k!}\sum_{i=0}^{j-1}S(j,j-i)(k)_{j-i}\\
&=\frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)e^{-(n+p)x}\sum_{k=0}^{\infty}(k)_{j-i}\frac{((n+p)x)^k}{k!}\\
&=\frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)((n+p)x)^{j-i}e^{-(n+p)x}\sum_{k=j-i}^{\infty}\frac{((n+p)x)^{k-j+i}}{(k-j+i)!}\\
&=\frac{1}{n^j}\sum_{i=0}^{j-1}S(j,j-i)((n+p)x)^{j-i}.
\end{aligned}$$

□

**Application 3.1.** For checking the correctness of the relation (3.12) in Proposition 3.2, we present the first four cases concerning computation of the test functions by Szász-Mirakjan-Schurer operators.

Case 1. $j=1$

$$S_{n,p}(e_1;x)=\frac{1}{n}S(1,1)(n+p)x=\frac{(n+p)x}{n}=x+\frac{px}{n}.$$

Case 2. $j = 2$

$$S_{n,p}(e_2;x) = \frac{1}{n^2}\sum_{i=0}^{1} S(2,2-i)((n+p)x)^{2-i} = \frac{(n+p)^2x^2}{n^2} + \frac{(n+p)x}{n^2}$$
$$= x^2 + \frac{\left(2np+p^2\right)x^2 + (n+p)x}{n^2}.$$

Case 3. $j = 3$

$$S_{n,p}(e_3;x) = \frac{1}{n^3}\sum_{i=0}^{2} S(3,3-i)((n+p)x)^{3-i} = \frac{(n+p)^3x^3}{n^3} + \frac{3(n+p)^2x^2}{n^3} + \frac{(n+p)x}{n^3}$$
$$= x^3 + \frac{\left(3n^2p+3np^2+p^3\right)x^3 + 3(n+p)^2x^2 + (n+p)x}{n^3}.$$

where $S(3,2) = 2\cdot S(2,2) + S(2,1) = 3$.

Case 4. $j = 4$

$$S_{n,p}(e_4;x) = \frac{1}{n^4}\sum_{i=0}^{3} S(4,4-i)((n+p)x)^{4-i} =$$
$$\frac{(n+p)^4x^4}{n^4} + \frac{6(n+p)^3x^3}{n^4} + \frac{7(n+p)^2x^2}{n^4} + \frac{(n+p)x}{n^4}$$
$$= x^4 + \frac{\left(4n^3p+6n^2p^2+4np^3+p^4\right)x^4 + 6(n+p)^3x^3 + 7(n+p)^2x^2 + (n+p)x}{n^4},$$

where $S(4,2) = 2\cdot S(3,2) + S(3,1) = 7$ and $S(4,3) = 3\cdot S(3,3) + S(3,2) = 6$.

The computation of higher order moments is tedious, but rather mechanical work. Using the Proposition 2.1, once we know the moments of low orders, we get the ones of higher order. So, it follows the following:

**Lemma 3.1.** *The Szász-Mirakjan-Schurer operators satisfy*

$$S_{n,p}(e_1 - x;x) = \frac{px}{n},$$
$$S_{n,p}\left((e_1-x)^2;x\right) = \frac{(px)^2 + (n+p)x}{n^2},$$
$$S_{n,p}\left((e_1-x)^3;x\right) = \frac{(px)^3 + 3p(n+p)x^2 + (n+p)x}{n^3},$$
$$S_{n,p}\left((e_1-x)^4;x\right) = \frac{(px)^4 + 6p^2(n+p)x^3 + (3n+7p)(n+p)x^2 + (n+p)x}{n^4}.$$

*Proof.* Taking into account the Proposition 2.1 and the Application 3.1 the computation of the moments up to the fourth moment is established. □

Now, using the construction form preliminaries, we assume that $I = J = [0,+\infty)$, $E(I) = C_2[0,+\infty)$, $F(J) = C[0,+\infty)$, the role of $n$ is played by $n+p$, then the functions $\varphi_{n+p,k} : [0,+\infty) \to \mathbb{R}$ be defined by $\varphi_{n+p,k}(x) := s_{n,p,k}(x)$, for any $x \in [0,+\infty)$, any $n,k \in \mathbb{N}_0$, $n \neq 0$ and the functionals $A_{n+p,k} : C_2[0,+\infty) \to \mathbb{R}$ be defined by $A_{n+p,k}(f) := f\left(\frac{k}{n}\right)$, for any $n,k \in \mathbb{N}_0$, $n \neq 0$.

In this way we get the Szász-Mirakjan-Schurer operators.

**Lemma 3.2.** *For any $x \in [0, +\infty)$ and any $n \in \mathbb{N}$, the following hold*
*i)* $T^*_{n,0}(S_{n,p}; x) = 1$,
*ii)* $T^*_{n,1}(S_{n,p}; x) = px$,
*iii)* $T^*_{n,2}(S_{n,p}; x) = (px)^2 + (n+p)x$,
*iv)* $T^*_{n,3}(S_{n,p}; x) = (px)^3 + 3p(n+p)x^2 + (n+p)x$,
*v)* $T^*_{n,4}(S_{n,p}; x) = (px)^4 + 6p^2(n+p)x^3 + (3n+7p)(n+p)x^2 + (n+p)x$.

*Proof.* Taking into account the fact that Szász-Mirakjan-Schurer operators reproduce constants, the relation (2.6), Application 3.1 and Lemma 3.1, it follows the identities i)–v). □

**Lemma 3.3.** *For any $x \in [0, +\infty)$, the following*

$$\lim_{n\to\infty} T^*_{n,0}(S_{n,p}; x) = 1, \tag{3.14}$$

$$\lim_{n\to\infty} \frac{T^*_{n,2}(S_{n,p}; x)}{n} = x, \tag{3.15}$$

$$\lim_{n\to\infty} \frac{T^*_{n,4}(S_{n,p}; x)}{n^2} = 3x^2, \tag{3.16}$$

*hold, then there exists $n(0) \in \mathbb{N}$, so that*

$$T^*_{n,0}(S_{n,p}; x) = 1 = k_0, \tag{3.17}$$

$$\frac{T^*_{n,2}(S_{n,p}; x)}{n} \le b + 1 = k_2, \tag{3.18}$$

$$\frac{T^*_{n,4}(S_{n,p}; x)}{n^2} \le 3b^2 + 1 = k_4, \tag{3.19}$$

*for any $x \in K = [0, b]$, $b > 0$, any $n \in \mathbb{N}$, $n \ge n(0)$.*

*Proof.* The relation (3.14)–(3.16) follow immediately from Lemma 3.2, while (3.17)–(3.19) follow from (3.14)–(3.16) by taking the definition of the limit into account. □

**Theorem 3.2.** *Let $f \in C_2[0, +\infty)$ be a function. If $x \in [0, +\infty)$ and $f$ is $s$ times differentiable in a neighborhood of $x$, then*

$$\lim_{n\to\infty} S_{n,p}(f; x) = f(x), \tag{3.20}$$

*for $s = 0$;*

$$\lim_{n\to\infty} n\left(S_{n,p}(f; x) - f(x)\right) = pxf^{(1)}(x) + \frac{1}{2}xf^{(2)}(x), \tag{3.21}$$

*for $s = 2$;*

$$\begin{aligned}\lim_{n\to\infty} n^2 \left(S_{n,p}(f; x) - f(x) - \frac{px}{n}f^{(1)}(x) - \frac{(px)^2 + (n+p)x}{2n^2}f^{(2)}(x)\right)& \\ = \frac{3px^2 + x}{6}f^{(3)}(x) + \frac{x^2}{8}f^{(4)}(x),&\end{aligned} \tag{3.22}$$

*for $s = 4$.*
*Assume that $f$ is $s$ times differentiable on $[0, +\infty)$, then the convergence from* (3.20)–(3.22) *is uniform, on any compact interval $K = [0, b] \subset [0, +\infty)$. Moreover, we get for any $x \in K$ and any $n \in \mathbb{N}$, $n \ge n(0)$*

$$|S_{n,p}(f; x) - f(x)| \le (b+2)\,\omega_1\left(f, \frac{1}{\sqrt{n}}\right),$$

*for* $s = 0$ *and*

$$n\left|S_{n,p}(f;x)-f(x)-\frac{px}{n}f^{(1)}(x)-\frac{(px)^2+(n+p)x}{2n^2}f^{(2)}(x)\right| \leq \frac{3b^2+b+2}{2}\,\omega_1\left(f^{(2)},\frac{1}{\sqrt{n}}\right),$$

*for* $s = 2$.

*Proof.* It follows from Theorem 2.1, with $\alpha_0 = 0$, $\alpha_2 = 1$ and $\alpha_4 = 2$, taking into account Lemma 3.2 and Lemma 3.3. □

**Remark 3.3.** The above theorem generalizes the asymptotic behavior of Szász-Mirakjan-Schurer operators. From among particular cases presented, we recover formula (3.21), which represents a Voronovskaja type formula for twice continuously differentiable functions, in the sense of Voronovskaja formula [14] and in the case when $p = 0$, we get the asymptotic behavior for the classical Mirakjan-Favard-Szász operators, (see [6] or [8]).

Concerning quantitative form of Voronovskaja result, in terms of the modulus of continuity, we get estimates for continuously function, respectively for twice continuously differentiable functions.

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