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Ulam-Hyers stability for operatorial inclusions

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ABSTRACT. The purpose of the work is to present some Ulam-Hyers stability results for the coincidence point problem associated to single-valued and multi-valued operators. As an application, an Ulam-Hyers stability theorem for a differential inclusion.

1. ULAM-HYERS STABILITY FOR COINCIDENCE EQUATIONS

Let (X, d) be a metric space and $f : X \to X$ an operator. We denote by

$$F_f := \{x \in X | f(x) = x\},$$

the fixed point set of the operator f. By definition, f is weakly Picard operator if the sequence $(f^n(x))_{n\in\mathbb{N}}$, of successive approximations converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f. For example, self Caristi type operators and self graphic contractions on complete metric spaces are examples of weakly Picard operators.

If f is weakly Picard operator then we consider the operator $f^\infty : X \to X$ defined by $f^\infty(x) := \lim\limits_{n\to\infty} f^n(x)$. It is clear that $f^\infty(X) = F_f$. Moreover, f^∞ is a set retraction of X to F_f.

If f is weakly Picard operator and $F_f = \{x^*\}$, then by definition f is a Picard operator. In this case f^∞ is the constant operator, $f^\infty(x) = x^*$, for all $x \in X$. Self Banach contractions, Kannan contractions and Ciric-Reich-Rus contractions on complete metric spaces are nice examples of Picard operators.

The following concepts are important in our consideration, see [6].

Definition 1.1. Let $f : X \to X$ be a weakly Picard operator and $c > 0$ a real number. By definition the operator f is c-weakly Picard operator if

$$d(x, f^\infty(x)) \le cd(x, f(x)), \text{ for all } x \in X.$$

Example 1.1. Let (X, d) be a complete metric space and $f : X \to X$ an operator with closed graphic. We suppose that f is a graphic α-contraction, i.e.,

$$d(f^2(x), f(x)) \le \alpha d(x, f(x)), \text{ for all } x \in X.$$

Then f is a c-weakly Picard operator, with $c = \dfrac{1}{1-\alpha}$.

Definition 1.2. Let (X, d) be a metric space and $f : X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{1.1}$$

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is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, f(y)) \leq \varepsilon \tag{1.2}$$

there exists a solution x^* of the equation (1.1) such that

$$d(y^*, x^*) \leq c_f \varepsilon.$$

The following abstract result was proved in [6].

Lemma 1.1. *If f is a c-weakly Picard operator, then the fixed point equation* (1.1) *is Ulam-Hyers stable.*

More generally, in [6] the following concept was introduced.

Definition 1.3. ([6]) Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. An operator $f : X \to X$ is said to be a ψ-weakly Picard operator if it is nonself weakly Picard operator and

$$d(x, f^\infty(x)) \leq \psi(d(x, f(x))), \text{ for all } x \in X.$$

In the case that $\psi(t) := ct$ (with $c > 0$), for each $t \in \mathbb{R}_+$, we say that f is c-weakly Picard operator.

Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ two operators. Let us consider the following coincidence point problem

$$f(x) = g(x) \tag{1.3}$$

We denote by $C(f, g)$ the set of coincidence points of f and g.

Definition 1.4. ([6]) Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ be two operators. The coincidence problem (1.3) is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution u^* of the inequality

$$\rho(f(u), g(u)) \leq \varepsilon \tag{1.4}$$

there exists a solution x^* of (1.3) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$ then the coincidence point (1.3) is said to be Ulam-Hyers stable.

Definition 1.5. ([6]) Let (X, d) and (Y, ρ) be two metric spaces. Then the operators $f, g : X \to Y$ form a ψ-weakly Picard pair, denoted by $[f, g]$ if $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ and there exists an operator $p : X \to X$ such that:

(i) p is a weakly Picard operator;
(ii) $Fix(p) = C(f, g)$;
(iii) $d(x, f^\infty(x)) \leq \psi(\rho(f(x), g(x)))$, for each $x \in X$.

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the operators $f, g : X \to Y$ form a c-weakly Picard pair.

A result on Ulam-Hyers stability of a coincidence point problem is the following theorem.

Theorem 1.1 ([6]). *Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ be two operators such that $[f, g]$ forms a ψ-weakly Picard pair (respectively a c-weakly Picard pair). Then the coincidence point problem* (1.3) *is generalized Ulam-Hyers stable (respectively Ulam-Hyers stable).*

Another result of this type, useful for applications, is the following theorem.

Theorem 1.2. *Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ be two operators. Suppose that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0, $\psi(0) = 0$ and there exists an operator $p : X \to X$ such that:*

(i) p is a ψ-weakly Picard operator;
(ii) Fix(p)=C(f,g);
(iii) $d(x, p(x)) \leq \rho(f(x), g(x))$ for each $x \in X$.

Then the coincidence point problem (1.3) *is generalized Ulam-Hyers stable (respectively Ulam-Hyers stable).*

Proof. Let $\varepsilon > 0$ and $u^* \in X$ be a solution of (1.4), i.e. $\rho(f(u^*), g(u^*)) \leq \varepsilon$. Then by (iii), for $u^* \in X$ we have $d(u^*, p(u^*)) \leq \rho(f(u^*), g(u^*))$.

Since p is a ψ-weakly Picard operator, we get that

$$d(x, p^\infty(x)) \leq \psi(d(x, p(x))), \text{ for each } x \in X.$$

If we denote $x^* := p^\infty(u^*)$, then by (i) and (iii), we obtain that $x^* \in C(f, g)$ and

$$d(u^*, x^*) = d(u^*, p^\infty(u^*)) \leq \psi(d(u^*, p(u^*))) \leq \psi(\rho(f(u^*), g(u^*))) \leq \psi(\varepsilon).$$ □

We will present now a consequence of the above abstract result.

The following auxiliary lemma is quite obvious.

Lemma 1.2. *Let X, Y be two nonempty sets and let $f, g : X \to Y$ be two operators. Suppose that f (respectively g) is onto. Then $C(f, g) = Fix(p)$, where $p := f^{-1} \circ g$ (respectively $p := g^{-1} \circ f$).*

By Lemma 1.2 and the above theorems we get the following result.

Theorem 1.3. *Let (X, d) and (Y, ρ) be two metric spaces and $f, g : X \to Y$ be two operators such that:*

(i) f is onto;
(ii) $f^{-1} \circ g$ is an a-contraction;
(iii) for each $x \in X$ we have $d(x, f^{-1}(g(x))) \leq \rho(f(x), g(x))$.

Then the coincidence point problem (1.3) *is Ulam-Hyers stable.*

Proof. By (i) and (ii) we get that $p := f^{-1} \circ g$ is a c-weakly Picard operator with $c := \frac{1}{1-a}$. Moreover, by Lemma 1.2 we have that $Fix(p) = C(f, g)$. By (iii) we obtain that the condition (iii) in Theorem 1.2 holds. The conclusion follows now by Theorem 1.2, □

Example 1.2. If $f, g : [0, 1] \to [0, 3]$ are given by $f(x) = 3x$, $g(x) = 2x$, for each $x \in [0, 1]$, then f is onto, $f^{-1} \circ g : [0, 1] \to [0, 1]$, given by $(f^{-1} \circ g)(x) = \frac{2x}{3}$ is $\frac{2}{3}$-contraction and $C(f, g) = \{0\}$.

Notice also that for each $x \in [0, 1]$ there exists $y \in [0, 1]$ such that $f(y) = g(x)$ and $|x - y| \leq \rho(f(x), g(x))$. We have $g(x) = 2x \in [0, 2]$ and $f(y) = 3y \in [0, 2] \Rightarrow y \in \left[0, \frac{2}{3}\right] \subset [0, 1]$. Thus, in this case, the coincidence point problem (1.3) is Ulam-Hyers stable.

We will present now a result on Ulam-Hyers stability for the case of Goebel coincidence theorem.

Theorem 1.4. *Let $A \neq \emptyset$ be an arbitrary set and let (M, d) be a metric space. Let $S, T : A \to M$ such that $S(A) \subset T(A)$ and $(T(A), d)$ is a complete subspace of M. Suppose that exists $0 \leq k < 1$ such that $d(Sx, Sy) \leq kd(Tx, Ty)$, for all $x, y \in A$. Then:*

a) $C(S, T) \neq \emptyset$ (Goebel's Theorem, see [4]*);*

b) If additionally:

$$d(y, S(T^{-1}(y))) \leq d(Ty, Sy), \text{ for all } y \in T(A), \tag{1.5}$$

then the coincidence point problem (1.3) *is Ulam-Hyers stable.*

Proof. a) Let $f := S \circ T^{-1}$. The proof is organized in several steps. We prove:

i) f is a singlevalued operator on $T(A)$;

Let $y_1, y_2 \in f(x)$. We get $y_1 \in S(T^{-1}(x))$ and $y_2 \in S(T^{-1}(x))$. So exists $u_1, u_2 \in T^{-1}(x)$ such that $y_1 = S(u_1)$ and $y_2 = S(u_2)$. Because $u_1, u_2 \in T^{-1}(x)$ we have $T(u_1) = x$ and $T(u_2) = x$. Then we have:

$$d(y_1, y_2) = d(Su_1, Su_2) \leq kd(Tu_1, Tu_2) = 0.$$

So $y_1 = y_2$ and thus $f(x)$ is a single point.

ii) $f : T(A) \to T(A)$;

Let $x \in T(A)$. Then exists $a \in A$ such that $x = T(a)$. So we have $a \in T^{-1}(x) \Rightarrow S(a) \subseteq S(T^{-1}(x)) \Rightarrow S(a) \subseteq f(x)$. Since f is a siglevalued operator we get $S(a) = f(x) \Rightarrow f(x) = S(a) \subseteq S(A) \subseteq T(A)$.

iii) $f : T(A) \to T(A)$ is k-contraction;

Let $x_1, x_2 \in T(A)$ and $u_1, u_2 \in A$ such that $u_1 \in T^{-1}(x_1)$ and $u_2 \in T^{-1}(x_2)$. Then we have: $d(f(x_1), f(x_2)) = d(S(T^{-1}(x_1)), S(T^{-1}(x_2))) = d(Su_1, Su_2) \leq kd(Tu_1, Tu_2) = kd(x_1, x_2)$. So f is a k-contraction.

iv) we apply Banach contraction principle for f and thus there exists a unique $y^* \in T(A)$ such that $y^* = f(y^*) = S(T^{-1}(y^*))$.

Let $x^* = T^{-1}(y^*) \Rightarrow y^* = T(x^*)$. We get $y^* = S(x^*)$. Then we have $S(x^*) = T(x^*) = y^*$.

b) Because $f : T(A) \to T(A)$, $f(x) = S \circ T^{-1}$ is a contraction then $Fix(f) = \{y^*\}$ and f is a Picard operator. So exists $c_f > 0$ such that for all $\varepsilon > 0$ and for all $u^* \in T(A)$ such that $d(u^*, f(u^*)) \leq \varepsilon$ we have $d(u^*, y^*) \leq \frac{1}{1-k} \cdot \varepsilon$.

We proof that the coincidence point problem is Ulam-Hyers stable.

Let $\varepsilon > 0$ and $x \in A$ such that $d(Tx, Sx) \leq \varepsilon$. Then we take account (1.5) we have $d(u^*, f(u^*)) \leq d(Tu^*, Su^*) \leq \varepsilon$. So we get

$$d(u^*, y^*) = d(u^*, f(y^*)) = d(f(y^*), f(u^*)) + d(f(u^*), u^*) = kd(u^*, y^*) + \varepsilon.$$

Then $d(u^*, y^*) = \frac{\varepsilon}{1-k}$. □

2. Ulam-Hyers stability for operatorial inclusions

The aim of this section is to prove an Ulam-Hyers stability result for a Cauchy problem associated to a differential inclusion of first order. We introduce first some notations and concepts.

Definition 2.6. (see [7], [5] and [6]) Let (X, d) be a metric space, and $F : X \to P_{cl}(X)$ be a multivalued operator. By definition, F is a multivalued weakly Picard operator if for each $x \in X$ and each $y \in F(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that:

(i) $x_0 = x, x_1 = y$;

(ii) $x_{n+1} \in F(x_n)$, for each $n \in \mathbb{N}$;
(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of F.

Let $F : X \to P(X)$ be a multi-valued weakly Picard operator. Denote

$$Graph(F) := \{(x, y) \in X \times X : x \in X \text{ and } y \in F(x)\}.$$

Then, we consider the multi-valued operator $F^\infty : Graph(F) \to P(Fix(F))$ defined by the following formula:
$F^\infty(x, y) := \{$ the set of all fixed points of F that are limits of a successive approximations sequence starting from $(x, y)\}$.

Definition 2.7. Let (X, d) be a metric space and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function which is continuous in 0 and $\psi(0) = 0$. Then $F : X \to P(X)$ is said to be a multi-valued ψ-weakly Picard operator if it is a multi-valued weakly Picard operator and there exists a selection $f^\infty : Graph(F) \to Fix(F)$ of F^∞ such that

$$d(x, f^\infty(x, y)) \leq \psi(d(x, y)), \text{ for all } (x, y) \in Graph(F).$$

If there exists $c > 0$ such that $\psi(t) := ct$ for each $t \in \mathbb{R}_+$, then we say that F is a multi-valued c-weakly Picard operator.

Definition 2.8. Let (X, d) be a metric space and $F : X \to P(X)$ be a multi-valued operator. The fixed point inclusion

$$x \in F(x), \quad x \in Y \tag{2.6}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequation

$$D(y, F(y)) \leq \varepsilon \tag{2.7}$$

there exists a solution x^* of the fixed point inclusion (2.6) such that

$$d(y^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) := ct$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (2.6) is said to be Ulam-Hyers stable.

The following abstract result has given in [2].

Lemma 2.3. *Let (X, d) be a metric space and $F : X \to P_{cp}(X)$ be a multi-valued ψ-weakly Picard operator. Then the fixed point inclusion (2.6) is generalized Ulam-Hyers stable.*

The aim of this section is, based on the above result, to prove an Ulam-Hyers stability theorem for a multi-valued Cauchy problem corresponding to a first order differential inclusion.

Let us consider the following multi-valued Cauchy problem:

$$\begin{cases} x'(t) \in F(t, x(t)), \text{ a.e. } t \in [a, b]; \\ x(a) = \alpha, \end{cases} \tag{2.8}$$

where $\alpha \in \mathbb{R}^n$ and $F : [a, b] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ is a multi-valued operator. We will denote by $\int_a^b F(s, x(s))ds$ (where $x : [a, b] \to \mathbb{R}^n$ is a given function) the multi-valued integral in Aumann' sense, see [1].

Definition 2.9. Let $F : [a, b] \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ a multi-valued operator, $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$. A function $\varphi : [a, T] \to \mathbb{R}^n$ is called solution to problem (2.8) if and only if $T \leq b$, φ is absolutely continuous on $[a, T]$ and satisfy the relations:

$$\begin{cases} \varphi'(t) \in F(t, \varphi(t)), & \text{a.e. on } [a, T]; \\ \varphi(a) = \alpha. \end{cases}$$

The equivalence between the above differential inclusion and an integral inclusion is given by the following lemma:

Lemma 2.4. *Let $I \subset \mathbb{R}$ an interval and $F : I \times \mathbb{R}^n \to P_{cp,cv}(\mathbb{R}^n)$ be an upper semi-continuous multi-valued operator. Then $x : I \to \mathbb{R}$ is a solution for differential inclusion*

$$x'(t) \in F(t, x(t)) \tag{2.9}$$

if and only if

$$x(t_2) \in x(t_1) + \int_{t_1}^{t_2} F(t, x(t))dt, \text{ for each } t_1, t_2 \in I. \tag{2.10}$$

Taking into account of Lemma 2.4 we deduce that the problem (2.8) is equivalent to an integral inclusion of Volterra type:

$$x(t) \in \alpha + \int_a^t F(s, x(s))ds, \quad t \in [a, b]. \tag{2.11}$$

The result with respect to the Ulam-Hyers stability of the Cauchy problem (2.8) is the following theorem.

Theorem 2.5. *Let $F : [a, b] \times \mathbb{R}^n \to P_{cl,cv}(\mathbb{R}^n)$ such that:*

(a) there exists an integrable function $M : [a, b] \to \mathbb{R}_+$ such that for each $u \in \mathbb{R}^n$ we have $F(s, u) \subset M(s)B(0, 1)$, a.e. $s \in [a, b]$;

(b) for each $u \in \mathbb{R}^n$, $F(\cdot, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is measurable;

(c) for each $u \in \mathbb{R}^n$, $F(\cdot, u) : [a, b] \to P_{cl,cv}(\mathbb{R}^n)$ is lower semi-continuous;

(d) there exists a continuous function $p : [a, b] \to \mathbb{R}_+$ such that for each $s \in [a, b]$ and each $u, v \in \mathbb{R}^n$ we have that:

$$H(F(s, u), F(s, v)) \leq p(s) \cdot |u - v|. \tag{2.12}$$

Then the following conclusions hold:

(i) there exists at least one solution for the Cauchy problem (2.8);

(ii) the Cauchy problem (2.8) is Ulam-Hyers stable, i.e. for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for each function $y \in C([a, b], \mathbb{R}^n)$ of the inequation

$$D_{\|\cdot\|_{\mathbb{R}^n}}(y(t), F(t, y(t))) \leq \varepsilon, \quad t \in [a, b],$$

wich satisfy the condition $y(a) = \alpha$ there exists a solution x of problem (2.8) such that

$$\|x(t) - y(t)\|_{\mathbb{R}^n} \leq c_\varepsilon \cdot \varepsilon, \text{ for each } t \in [a, b].$$

Proof. We consider the multi-valued operator $T : C([a, b], \mathbb{R}^n) \to P(C([a, b], \mathbb{R}^n))$ defined by

$$T(x) := \left\{ v \in C([a, b], \mathbb{R}^n) | v(t) \in \alpha + \int_a^t F(s, x(s))ds, t \in [a, b] \right\}.$$

Then (2.8) is equivalent to (2.11) and, by the above notation, equivalent to the fixed point inclusion

$$x \in T(x), \quad x \in C([a, b], \mathbb{R}^n). \tag{2.13}$$

The proof is organized in several steps:

1). $T(x) \in P_{cp}(C([a,b],\mathbb{R}^n))$.

From Theorem 2 in Rybinski [8] we have that for each $x \in C([a,b],\mathbb{R}^n)$ there exists $f(s) \in F(s,x(s))$, for all $s \in [a,b]$, such that $f(s)$ is integrable.

Then $v(t) := \alpha + \int_a^t f(s)ds$ has the property $v \in T(x)$. Moreover, from (a) and (b), via Theorem 8.6.3. in Aubin and Frankowska [1], we get that $T(x)$ is a compact set, for each $x \in C([a,b],\mathbb{R}^n)$.

2). T is a multi-valued contraction on $C([a,b],\mathbb{R}^n)$.

Notice first that one may suppose (without affecting the generality of the Lipschitz condition) that the inequality (2.12) is strict. Let $x_1, x_2 \in C([a,b],\mathbb{R}^n)$ and $v_1 \in T(x_1)$. Then $v_1(t) \in \alpha + \int_a^t F(s,x_1(s))ds$, $t \in [a,b]$. It follows that

$$v_1(t) \in \alpha + \int_a^t f_1(s)ds,\ t \in [a,b],\ \text{for some } f_1(s) \in F(s,x_1(s))ds,\ s \in [a,b].$$

From (d) we have $H(F(s,x_1(s)),F(s,x_2(s))) < p(s)\cdot|x_1(s)-x_2(s)|$. Thus there exists $w \in F(s,x_2(s)$ such that $|f_1(s)-w| \le p(s)\cdot|x_1(s)-x_2(s)|$, for $s \in [a,b]$.

Let as define $U : [a,b] \to P(\mathbb{R}^n)$, by

$$U(s) := \{w|\ |f_1(s)-w| \le p(s)\cdot|x_1(s)-x_2(s)|\}.$$

Since the multi-valued operator $V(s) := U(s) \cap F(s,x_2(s))$ is measurable, so exists $f_2(s)$ a selection for V, measurable (and hence integrable in s). Hence $f_2(s) \in F(s,x_2(s))$ and $|f_1(s)-f_2(s)| \le p(s)\cdot|x_1(s)-x_2(s)|$, for each $s \in [a,b]$.

Consider $v_2(t) = \alpha + \int_a^t f_2(s)$, $t \in [a,b]$. We denote by $\|\cdot\|_B$ a Bielecki-type norm in $C([a,b],\mathbb{R}^n)$, given by

$$\|x\|_B := \sup_{t\in[a,b]} (\|x(t)\|_{\mathbb{R}^n}\cdot e^{-\tau q(t)}),\ \text{where } q(t) := \int_a^t p(s)ds.$$

Then for each $t \in [a,b]$, we have:

$$|v_1(t)-v_2(t)| \le \int_a^t |f_1(s)-f_2(s)|ds \le \int_a^t p(s)|x_1(s)-x_2(s)|ds =$$
$$= \int_a^t p(s)e^{\tau q(s)}|x_1(s)-x_2(s)|e^{-\tau q(s)}ds \le \int_a^t p(s)e^{\tau q(s)}\|x_1-x_2\|_B ds =$$
$$= \frac{1}{\tau}\|x_1-x_2\|_B(e^{\tau q(t)}-e^{\tau q(a)}) \le \frac{1}{\tau}\|x_1-x_2\|_B e^{\tau q(t)}.$$

Thus we immediately get $\|v_1-v_2\|_B \le \frac{1}{\tau}\|x_1-x_2\|_B$. A similar relation can be obtained by interchanging the role of x_1 and x_2. By choosing now $\tau > 1$ we get that $H_{\|\cdot\|_B}(T(x_1),T(x_2)) \le \frac{1}{\tau}\|x_1-x_2\|_B$, which prove that T is a multi-valued contraction with constant $\alpha := \frac{1}{\tau}$. Hence, conclusion (i) follows by Covitz-Nadler's fixed point theorem [3].

For the second conclusion, let $\varepsilon > 0$ and $y \in C([a,b],\mathbb{R}^n)$ for which there exits $u \in C([a,b],\mathbb{R}^n)$ such that $u(t) \in \alpha + \int_a^t F(s,y(s))ds$, $t \in [a,b]$ and $\|u(t)-y(t)\|_{\mathbb{R}^n} \le \varepsilon$, for each $t \in [a,b]$. Notice that $\|\cdot\|_B \le \|\cdot\|_C \le \|\cdot\|_B e^{\tau q(b)}$, where $\|\cdot\|_C$ denotes the Cebâşev norm in $C([a,b],\mathbb{R}^n)$ defined by $\|x\|_C := \max\limits_{t\in[a,b]}(\|x(t)\|_{\mathbb{R}^n})$. Then we obtain that $\|u-y\|_B \le \|u-y\|_C \le \varepsilon$. Thus, $D_{\|\cdot\|_B}(y,T(y)) \le \varepsilon$. Moreover, since T is a multi-valued α-contraction with respect to $\|\cdot\|_B$, we obtain that T is a multi-valued c-weakly Picard operator with $c := \dfrac{1}{1-\alpha}$. The conclusion (ii) is a consequence of Lemma 2.3. Hence, there exists $c := \dfrac{1}{1-\alpha}$ and a solution x^* of the Cauchy problem (2.8) such that $\|y-x^*\|_B \le c\cdot\varepsilon$. Hence $|y(t)-x^*(t)| \le c\cdot e^{\tau q(b)}\cdot\varepsilon,$ for each $t\in[a,b]$. □

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